Qualified topological relations between spatial objects with possible vague shape

L. Bejaoui a,b; F. Pinet c; Y. Bedard a,b; M. Schneider c,d

a Department of Sciences géomatiques, Laval University, Quebec city, Que., Canada
b Industrial Research Chair in Geospatial Databases for Decision Support, Laval University, Quebec city, Que., Canada
c Cemagref-Clermont-Ferrand, France
d Department of Computer Sciences, Blaise-Pascal University, Clermont-Ferrand, France

First Published: July 2009
Qualified topological relations between spatial objects with possible vague shape

L. BEJAOUI*†§¶, F. PINET§, Y. BEDARD†‡ and M. SCHNEIDER§¶
†Department of Sciences géomatiques, Laval University, Quebec city, Que., Canada
‡Industrial Research Chair in Geospatial Databases for Decision Support, Laval University, Quebec city, Que., Canada
§Cemagref-Clermont-Ferrand, France
¶Department of Computer Sciences, Blaise-Pascal University, Clermont-Ferrand, France

(Received 10 December 2007; in final form 27 February 2008)

Broad boundary is generally used to replace one-dimensional boundary for spatial objects with vague shape. For regions with broad boundary, this concept should respect both connectedness and closeness conditions. Therefore, some real configurations, like regions with partially broad boundary (e.g. lake with rocky and swamp banks), are considered invalid. This paper aims to represent different levels of shape vagueness and consider them during the identification of topological relations. Then, an object with vague shape is composed by two crisp components: a minimal extent and a maximal extent. Topological relations are identified by applying the 9-Intersection model for the subrelations between the minimal and maximal extents of objects involved. Four subrelations are then represented through a 4-Intersection matrix used to classify the topological relations. For regions with broad boundary, 242 relations are distinguished and classified into 40 clusters. This approach supports an adverbial expression of integrity constraints and spatial queries.

Keywords: Spatial objects with vague shape; Topological relations; Partial shape vagueness; Spatial query; Integrity constraints; Spatial data integration

1. Introduction

To satisfy the requirements of several categories of users, Geographic Information Systems (GIS) and spatial databases provide tools to store, retrieve, analyze and display spatial data. Ensuring their usability requires controlling the spatial data quality, which can be degraded by several types of imperfections. Several approaches (Smithson 1989; Fisher 1999; Mowrer 1999; Duckham et al. 2001) proposed different categorizations of data imperfections that are generally caused by the complexity of reality and limitations of the instruments and processes used in the measurements (Bédard 1987). Moreover, inappropriate spatial data representations can also be another source of data quality degradation (Yazici et al. 2001; Shu et al. 2003; Dilo 2006). Spatial reality is generally forced to be represented by crisp spatial object types (i.e. points, lines and regions), whereas the shapes of many spatial objects are inherently vague (e.g. forest stand, pollution zone, valley or lake). Shape

*Corresponding author. Email: lotfi.bejaoui.1@ulaval.ca
vagueness occurs when it is difficult to distinguish the boundary (e.g. regions with broad boundary) and/or the interior (e.g. broad points or lines with broad interior) of an object’s geometry from other spatial objects of the neighborhood. Using crisp spatial object types to represent spatial objects with vague shape entails a clear gap between the spatial reality and its formal representation in databases and GIS (Cheng et al. 2001; Yazici et al. 2001).

Pertinent solutions were found to overcome the 'classical' sources of spatial data quality degradation (Bédard 1987; Goodchild 1995; Guptill and Morisson 1995; Ubeda and Egenhofer 1997; Frank 2001; Van Oort 2006; Devillers et al. 2007; Pinet et al. 2007). Several approaches (Burrough and Frank 1996; Cohn and Gotts 1996; Clementini and Di Felice 1997; Erwig and Schneider 1997; Schneider 2001; Tang 2004; Pfoser et al. 2005; Dilo 2006) have studied specificities of objects with vague shape to determine their appropriate representation. A review of the literature in this domain stresses that current GIS and spatial database systems do not offer the specific structure to formally represent this type of object (as pointed by Clementini and Di Felice 1997 10 years ago). With regard to this problem, researchers are increasingly motivated to model shape vagueness in order to: (1) reduce the gap between the geographic reality and the spatial models (Cohn and Gotts 1996); (2) provide formal modeling tools to represent shape vagueness (Yazici et al. 2001); (3) specify spatial queries involving spatial objects with vague shape (Erwig and Schneider 1997). In the same way, the spatial data integration requires the extraction of heterogeneous representations of the same objects from different data sources. The main difficulty lies in choosing one of them when no information exists about their quality (Rodriguez 2005). Using a spatial model that supports shape vagueness, it becomes possible to merge different representations in such a way that the integration result looks like an object with vague shape. For example, Figure 1 shows a spatial object that has a representation A in a first source and a representation B in a second one. The integration result can correspond to one geometry with vague shape made up of A and B (Figure 1). The intersection of A and B corresponds to the certain part (i.e. the part that exists in both representation A and representation B) or the minimal extent of the spatial object. However, the union is the maximal extent that the object can fill; it groups the certain and the uncertain parts (i.e. a geometry part is uncertain when it does not exist in all candidate representations for the integration) of the geometry. Indeed, there are strong and different motivations to present pertinent solutions in order to adequately model the shape vagueness.

To model objects with vague shape, researchers were firstly inspired by the modeling of crisp spatial objects. In point-set topology (Egenhofer and Herring 1990), crisp spatial objects are typically decomposed into three mutually disjoint topological invariants: an interior, a boundary and an exterior. Several approaches (Clementini and Di Felice 1997; Tang 2004; Reis et al. 2006) extend the crisp models by identifying other topological invariants for the objects with vague shape. For

![Figure 1](image-url)
example, Clementini and Di Felice (Clementini and Di Felice 1997) distinguish three topological invariants for regions with broad boundary: an interior, a broad boundary (i.e. a two–dimensional boundary) and an exterior. In this approach, the shape vagueness is correlated to the broad boundary, which should respect the closeness and the connectedness conditions (Clementini and Di Felice 1997; Tang 2004). Thus, any representation that does not verify these conditions is considered invalid. Nonetheless, the shape vagueness can also characterize only some parts of an object’s geometry. For example, Figure 2 shows a lake surrounded by crisp rocky banks on one side and swamp ones on the other side at the same time (Figure 2). We denote this kind of feature as objects with partially vague shape that cannot be represented by existing models. Then, the main questions are: How is it possible to define an exact model where different levels of shape vagueness could be considered? How can we retain this expressivity during the specification of topological relations between such objects?

The first objective of this paper is to allow the representation of three levels of shape vagueness: crispness, partial shape vagueness and complete shape vagueness. Modeling objects with vague shape requires a framework for identifying topological relations. The second objective is to consider the different levels of vagueness in the identification of topological relations between objects with vague shape. In several studies (Clementini and Di Felice 1997; Tang 2004; Reis et al. 2006), topological relations can be identified by enumerating the intersections between the topological invariants of the objects with vague shape involved. For each model, the number of relations depends upon the number of topological invariants. In this work, we look for an expressive model in which it is possible to specify the vagueness level of the topological relation instances. We think that it would be pertinent for the user to know whether objects are weakly or strongly disjoint. Accordingly, the third specific objective of this work is to classify the topological relations according to their vagueness level.

The remainder of the paper is organized as follows. In Section 2, we present previous works on the modeling of objects with vague shape and their topological relations. In Section 3, we present three basic types of spatial objects with vague shape: regions with broad boundary, lines with vague shape and broad points. Then, Section 4 gives a proposition based on the 9-Intersection model (Egenhofer and Herring 1990) in order to identify the topological relations among objects with vague shape. The model is applied to regions with broad boundary, and their topological relations are studied in detail in the Appendix. As a result of this approach, 242 relations can be distinguished through a 4-Intersection matrix. Section 5 proposes a hierarchical clustering of topological relations between regions with broad boundary, and Section 6 explains how to use our approach to express spatial queries and integrity constraints. In Section 7, our model is compared with existing exact approaches (Cohn and Gotts 1996; Clementini and Di Felice 1997; Tang 2004). Finally, Section 8 presents our conclusions and discusses future research.
2. Previous work

2.1 Spatial vagueness

According to previous studies (Erwig and Schneider 1997; Hazarika and Cohn 2001; Pfoser et al. 2005), spatial vagueness can characterize the position and/or shape of an object’s spatial extent. From this perspective, the shape vagueness refers to the difficulty of distinguishing one object’s shape from its neighborhood. Shape vagueness is an intrinsic property of an object that certainly has an extent in a known position but cannot or does not have a well-defined shape (Erwig and Schneider 1997). For example, a region has a vague shape when it is surrounded by broad instead of sharp boundaries. One could normally use the term ‘fuzziness’ to speak about ‘shape vagueness’ since it would correspond to the unclearness of an object shape as it is defined in a general ontology (i.e. to the definitions found in the Oxford and the Cambridge dictionaries). Nevertheless, in order to avoid confusion with the mathematical definition found in the specialized ontology of Fuzzy Set Theory (Zadeh 1965) which is used in several GIS-related papers (e.g. Altman 1987; Burrough 1989; Brown 1998; Schneider 2001), we have decided to use the expression ‘shape vagueness’. Accordingly, one must not confuse ‘fuzziness’ as defined in Fuzzy Set Theory with the concept of ‘shape vagueness’ as defined in the present paper.

Spatial vagueness can also characterize well-defined (or crisp) objects when there is uncertainty about objects’ positions despite their sharp shapes; we refer to this scenario as positional vagueness. Positional vagueness is a measurement imperfection related to the accuracy and precision of the instruments and processes used in the measurements (Mowrer 1999). Figure 3 shows this categorization of spatial vagueness into ‘shape vagueness’ and ‘positional vagueness’. In this paper, we only deal with the formal representation of spatial objects with vague shape and the topological relations between them.

In general, we distinguish between at least two categories of models used to represent spatial vagueness. In the first category, crisp spatial concepts are transferred and extended to formally express spatial vagueness; we speak about exact models (Cohn and Gotts 1996; Clementini and Di Felice 1997; Erwig and Schneider 1997) as explained in the next section. In the second category, three principal mathematical theories are generally used: (1) models based on the Fuzzy Logic (Zadeh 1965) (e.g. Altman 1987; Burrough 1989; Brown 1998; Schneider 2001; Tang 2004; Hwang and Thill 2005; Dilo 2006), which can be used to represent continuous phenomena such as temperature; (2) models based on rough sets (e.g. Ahlqvist et al. 1998; Worboys 1998), which represent the objects with vague shape as a pair of approximations (upper and lower approximations); (3) models based on probability theory (e.g. Burrough and Frank 1996; Pfoser et al. 2005), which is...
principally used to model errors of positions and attributes. In the next section, we present works that formally define objects with vague shape.

2.2 Formal definitions of objects with vague shape

2.2.1 Definitions based on exact models. The Egg–Yolk theory (Cohn and Gotts 1996) is an extension of the RCC (Region Connection Calculus) model (Randell and Cohn 1989; Cohn et al. 1997), it introduces the concept of regions with broad boundary. In fact, a region with broad boundary is composed of two crisp regions. The inner region is called ‘yolk’ (i.e. the certain part of the geometry), and it is surrounded by an outer region called ‘white’ (i.e. the broad boundary or the uncertain part of the geometry). The union of the ‘yolk’ and the ‘white’ corresponds to the ‘egg’. Because points and lines are ignored in the RCC model, their shape vagueness is not studied in this approach. In addition, regions with broad boundary with empty ‘yolk’ or empty ‘egg’ are not admitted. Thus, the crisp regions cannot be represented through the Egg–Yolk theory.

In the same way, Clementini and Di Felice (Clementini and Di Felice 1997) define the regions with broad boundary based on the point-set topology (Egenhofer and Herring 1990). Thus, a region with broad boundary $A$ is made up of two simple crisp regions $A_1$ and $A_2$, where $A_1 \subseteq A_2$. The broad boundary represents the shape vagueness, and it is equal to the closure of their difference $\Delta A = A_1 - A_2$. In this approach, $A_1$ and $A_2$ should be topologically consistent; this means that they should be bounded, regular and closed sets in $\mathbb{R}^2$ (Clementini and Di Felice 1997). Moreover, the authors distinguish two kinds of lines with vague shape: broad lines (i.e. all of the line is large) and lines with broad boundary (i.e. the line’s endpoints are ill-defined). Tang (Tang 2004) extends this approach and proposes a more detailed formal definition of regions with broad boundary, where he distinguishes four mutually disjoint components: the interior, the boundary’s interior, the boundary’s boundary and the exterior (Figure 4).

The condition $A_1 \subseteq A_2$ in the paper of Clementini and Di Felice (1997) does not exist in the paper of Erwig and Schneider (1997). In fact, Erwig and Schneider (Erwig and Schneider 1997) are interested in another kind of vagueness, where a region with vague shape is a composed geometry. The geometry’s components belong to a pair of subsets. First, the kernel subset contains the subregions definitely belonging to the region with vague shape. Second, the boundary subset contains the subregions possibly belonging to the region with vague shape. Likewise, the points with vague shape and the lines with vague shape are respectively defined as a pair of subsets of points and lines. Crisp spatial objects can be expressed through this model when the boundary subset is empty. Figure 5 gives an example of a region with vague shape.
2.2.2 Models based on mathematical approaches. Roy and Stell (Roy and Stell 2001) deal with indeterminacy, which is defined as knowledge vagueness. They define an indeterminate region through rough sets (Pawlak 1994). An indeterminate region is composed of a lower and an upper approximation. The difference between these approximations represents the broad part of the region. When this difference is empty, the region is crisp because the two approximations are equal (Roy and Stell 2001).

Fuzzy Set Theory is also used to represent objects with vague shape (called fuzzy spatial objects in this category of models) (Robinson and Thongs 1986; Altman 1987; Burrough 1989; Zhan 1997; Schneider 2001; Bordogna and Chiesa 2003; Tang et al. 2003; Tang 2004; Li and Li 2004; Dilo 2006; Verstraete et al. 2007). Zhan (Zhan 1997) and Dilo (Dilo 2006) interpret a fuzzy spatial object as a fuzzy subset. In the paper of Zhan (1997), the membership function of a fuzzy spatial object is decomposed to $a$–cuts (an $a$–cut is the crisp set of points with a membership value $a$ or higher (e.g. Godjjevac 1999)) in order to facilitate its interpretation. Accordingly, Somodevilla and Petry (Somodevilla and Petry 2003) represent a fuzzy region as a set of $a$–cuts nested around the minimum bounding rectangle enclosing the region. Schmitz and Morris (Schmitz and Morris 2006) propose a fuzzy model (in the sense of Fuzzy Logic) based on the $a$–cut notion in order to model fuzzy regions. They use this notion to describe the internal structure of a fuzzy boundary because one of limitations of fuzzy approaches is the interpretation and the use of a membership function of a fuzzy set. The definition of fuzzy regions (Schmitz and Morris 2006) assumes that the boundary is broad everywhere and an $a$–cut surrounds uniformly the core. We find this hypothesis little realistic because regions can have a broad boundary in some locations and a sharp boundary in some others (e.g. a lake with crisp rocky banks on one side and swamp ones on the other side). Then, the definition of $a$–cuts (Schmitz and Morris 2006) cannot be applied to such a common situation because it requires that an $a$–cut surrounds the core in the broad part and superposes the core’s boundary in the sharp part. It means that one $a$–cut could not have the same definition everywhere within a region with partially broad boundary. We would clarify that we do not discuss the limitations of the model presented in (Schmitz and Morris 2006) but we try to prove the importance of the distinction between ‘partially broad boundary’ and ‘completely broad boundary’ that we underline in the remainder of paper.

Bordogna and Chiesa (Bordogna and Chiesa 2003) adopt a Fuzzy Object-Oriented Data modeling approach in order to identify objects with broad boundary and their relations. Tang (Tang et al. 2003; Tang 2004) defines the objects with vague shape in two different ways. The first definition should respect the properties of the crisp topological space. The second definition should respect topological properties of a fuzzy topological space. In this context, the concept of fuzzy topology
is a generalization of crisp topology, in which the sets belonging to the universe are fuzzy (i.e. the membership value is \( z \), where \( 0 \leq z \leq 1 \)). Shi and Liu (Shi and Liu 2007) defined an interior operator and a closure operator in order to compute fuzzy spatial objects in a fuzzy topological space. Schneider (Schneider 2001) presents formal definitions of objects with vague shape through fuzzy subsets. For example, a region with broad boundary is defined as a generalization of a crisp region in which an arbitrary point of space has a partial membership to the region (Schneider 2001). In the same way, Bjørke (Bjørke 2004) proposed a method to compute the fuzzy boundaries of a region by associating for each point of the region a partial membership in both the interior and the boundary of the region.

In the next section, we highlight some approaches that study the specificities of the topological relations between spatial objects with broad boundary.

### 2.3 Topological relations between objects with vague shape

Topological relations are very important in GIS, especially in the specification of integrity constraints (Ubeda and Egenhofer 1997) and spatial queries. These relations are based only on the shape of objects and are totally independent of both the coordinates system and geometric transformations (Zhan and Lin 2003). The use of special shapes to express shape vagueness should imply a different approach for identifying their topological relations. In the crisp context, several models (Egenhofer and Herring 1990; Egenhofer and Franzosa 1991; Mark and Egenhofer 1994; Cohn et al. 1997) studied the specification of topological relations in GIS and spatial databases. These models are based on two main approaches: general point-set topology (Egenhofer and Herring 1990) and mereology1 (Varzi 2004). First, the 9-intersection model (Egenhofer and Herring 1990) is an extension of the 4-Intersection model (Egenhofer 1989). Based on point-set topology, the 9-Intersection model distinguishes eight relations between two simple regions (Disjoint, Meet, Overlap, Contains, Inside, Equal, Covers, Covered by), 36 between two simple lines, 19 between a simple region and a simple line, two between two points, three between a point and a simple line and three between a point and a simple region. In the context of mereology, a first model called RCC-5 (Randell and Cohn 1989) distinguished five topological relations among two simple crisp regions. This model was then extended to distinguish eight topological relations, and it became known as RCC-8 (Cohn et al. 1997).

In the shape vagueness context, topological relations can be specified by extending the RCC and 9-Intersection models (Cohn and Gotts 1996; Clementini and Di Felice 1997; Erwig and Schneider 1997; Roy and Stell 2001; Tang 2004). Erwig and Schneider (Erwig and Schneider 1997) use a three-valued logic to underline the shape vagueness. Then, an intersection between two topological invariants can be true, false or maybe (i.e. when the boundary subset participates in the relation). In the paper of Cohn and Gotts (1996), a topological relation between two Egg–Yolk regions \( A \) and \( B \) is specified through a \( 2 \times 2 \)-matrix that enumerates four subrelations: Egg\((A)–Egg(B)\), Egg\((A)–Yolk(B)\), Yolk\((B)–Egg(A)\) and Yolk\((A)–Yolk(B)\) (Figure 6). These four subrelations are those defined in the RCC-5 model: Partially Overlapping (PO), Proper Part (PP), Equal (E), Proper Part Inverse (PPI) and Distinct (D). With this approach, only 46 combinations correspond to the 46 possible topological relations drawn in (Cohn and Gotts 1996). Figure 6 presents the relation no. 15 in the paper of Cohn and Gotts (1996). The main advantage of this approach is its simplicity in identifying the topological relations. However, point, lines and crisp regions are not covered.
Clementini and Di Felice (Clementini and Di Felice 1997) studied approximate topological relations, which correspond to the topological relations between regions with broad boundary. In fact, they use a $3 \times 3$-matrix, in which the sharp boundary is replaced by a broad boundary. After considering a set of consistency rules for the matrices (i.e. 12 rules to eliminate each matrix that cannot be drawn) (Clementini and Di Felice 1997), only 44 relations can be distinguished and drawn among two simple regions with broad boundary. These relations can then be grouped into 17 clusters, for which a conceptual-neighborhood graph was drawn. This approach can be very interesting when it is necessary to coarsely identify the topological relations between regions with broad boundary. However, the model is not sufficient when the needs are more specific and the user has a clear idea regarding the relation between regions with broad boundary. For example, Figure 7 shows an example of two different relations absorbed by the same cluster and identified through the same matrix.

In the same way, Reis et al. (Reis et al. 2006) use Clementini’s model (Clementini and Di Felice 1997) in order to identify the topological relations between lines with vague shape. After considering the same 12 conditions defined in the paper of Clementini and Di Felice (1997), five topological relations are identified between two broad lines and 77 between two lines with broad boundary.

Tang (Tang 2004) presents an extension of the 9-Intersection model that identifies more topological relations than that of Clementini and Di Felice (1997). He identifies topological relations through a $4 \times 4$-matrix. Indeed, he distinguishes 152 topological relations and presents their correspondent matrices (see, the example, Figure 8). The absence of relation clustering is the main limitation of the model. The model becomes useless in practice, however, since it is very difficult to easily and intuitively distinguish all of these relations. Furthermore, Tang (Tang 2004) does not distinguish between the inner and outer boundary. Hence, many relations cannot be specified through a $4 \times 4$-matrix.

Fuzzy Set Theory is also used to identify the topological relations between objects with vague shape (Zhan 1997; Schneider 2001; Somodevilla and Petry 2003; Bjørke 2004; Du et al. 2005; Dilo 2006). A topological relation in Zhan (1997) is called $R$, i.e. a parameter used to replace the eight relations of the 4-Intersection model (Egenhofer 1989). For each pair of $\alpha$-cuts of considered regions, a subrelation $r$ is identified. Then, the possibility of the global relation $R$ is deduced from the number

Figure 6. Identification of topological relations in the work of Cohn and Gotts (1996).

![Figure 6](image6.png)

Figure 7. Identification of topological relations in the work of Clementini and Di Felice (1997).

![Figure 7](image7.png)
of subrelations $r$ that arise between the different $\alpha$-cuts. This approach is easy to use in practice, but it presents some complexity when the $\alpha$-cuts are non-uniformly distributed between 0 and 1. In the same way, Dilo (Dilo 2006) identifies six possible topological relations (i.e. *Disjoint*, *Touches*, *Crosses*, *Overlaps*, *Within* and *Equal*) between two spatial objects with vague shape. A topological relation is defined based on fuzzy operations (e.g. union, intersection, absolute difference and bounded difference) applied to fuzzy subsets that represent objects with vague shape. According to Dilo (Dilo 2006), many topological relations may exist at the same time with different *Truth degrees* (e.g. *Overlap*($A,B$) with the *Truth degree* $=0.2$; *Meet*($A,B$) with the *Truth degree* $=0.3$; *Disjoint*($A,B$) with the *Truth degree* $=0.5$). Du et al. (Du et al. 2005) proposed an extension of the 9-Intersection model in order to describe the fuzziness of topological relations caused by the fuzziness of spatial data. Shi and Liu (Shi and Liu 2007) consider two stages for modeling fuzzy topological relations between spatial objects: (1) to define and describe qualitatively; (2) to compute the fuzzy topological relations quantitatively using a quantitative approach such as Fuzzy Set Theory. In the same way, Bjørke (Bjørke 2004) uses a linguistic variable which gives an association to a crisp relation and a quantifier which indicates the strength of the topological relation computed using fuzzy operators. Fuzzy approaches provide quantitative evaluation of shape vagueness of spatial objects and relations. Accordingly, De Tré et al. (De Tré et al. 2004) extend *generalized constraints*, originally introduced by Zadeh (Zadeh 1965), to spatio-temporal data modeling to deal with partial satisfaction on integrity constraints involving fuzzy spatial objects. Then, integrity constraints can be partially respected because it can have any value between 0 and 1 (crisp constraint can be true or false).

Fuzzy models allow a description of the internal structure of broad parts of an object with vague shape. However, some quantitative hypotheses are generally required in order to define the membership functions either for computation of spatial objects or for evaluation of topological relations between them. It is a limitation of fuzzy approaches because selection of these hypotheses is not always based on perception studies or application dependent evaluations, but it is generally more or less arbitrary (Bjørke 2004). Additionally, fuzzy approaches are expensive in implementation and more adapted to raster format than vector format because gradual transition of the interior or the boundary of a fuzzy object can be shown through the strength computed for each pixel (Clementini 2005).

### 2.4 Problem statement

The exact models presented earlier (Cohn and Gotts 1996; Clementini and Di Felice 1997; Erwig and Schneider 1997; Tang 2004; Reis et al. 2006) have the advantage of explicitly distinguishing objects’ topological invariants. Through this discrete
viewpoint of space, the specification of topological relations can be improved (Clementini and Di Felice 1997). For these reasons, we propose an exact model in order to achieve objectives. Nevertheless, we think that the existing models do not distinguish between different levels of shape vagueness and are not sufficiently expressive to represent partial shape vagueness. In reality, a region with broad boundary is not always surrounded by a large boundary everywhere. For example, a lake’s boundary can be broad in some locations and sharp in some others. This situation cannot be represented by existing exact models, because the connectedness condition is violated. The same problem is present for lines. Only two cases of shape vagueness are distinguished in lines (cf. Section 2.2.1), but a line can easily have a partially broad interior independently of the boundary. Moreover, the studied models are not sufficiently expressive in terms of topological relations since there is no distinction between the inner and outer boundary for regions with broad boundary. Some works try to offer more expressivity by increasing the number of topological invariants (e.g. Tang 2004). Nevertheless, the absence of relation clustering limits their practical use. Indeed, the main research questions of our paper are the following:

1. How can we obtain more expressive definitions of the objects with vague shape through an exact model? How can we represent shape vagueness?
2. What are the topological relations between objects with vague shape? How is it possible to identify topological relations between objects that have different levels of shape vagueness? How can we formally identify these relations?
3. How can we classify the topological relations between regions with broad boundary in order to facilitate their use in practice? How could resulting clusters reflect the vagueness level of a topological relation?

3. Spatial objects with vague shape

In general, there is no agreement regarding the appropriate formal definition of spatial objects with vague shape, because shape vagueness can be interpreted in different ways. It is not the objective of this work to unify these interpretations. We are interested in proposing an expressive and easy definition of spatial objects with vague shape through an exact model. In our approach, we transfer the Egg–Yolk model into point-set topology context in order to both consider points and lines and permit the representation of objects with partially vague shape. In the literature, many expressions have been used to speak about shape vagueness of spatial objects. For example, Burrough and Frank (Burrough and Frank 1996) used the terms ‘objects with indeterminate boundaries’, Dilo (Dilo 2006) used the terms ‘vague spatial objects’ and Clementini and Di Felice (Clementini and Di Felice 1997) used ‘objects with large boundary’. We find these different expressions pertinent but they are not sufficiently expressive to cover the shape vagueness for a line’s interior or a point (i.e. a point does not have a boundary; it is composed by an interior). In other words, we make a distinction between ‘broad interior’ and ‘broad boundary’ that we consider as specializations of ‘shape vagueness’. This distinction is useful especially in the cases of lines and points. From this perspective, we distinguish three basic types of spatial objects with vague shape: broad points, lines with vague shape (i.e. lines with broad boundary, lines with broad interior or broad lines), and regions with broad boundary. Figure 9 shows our categorization of objects with vague shape.
Each object with vague shape is composed of \( n \) crisp object types (i.e. point, line and region) distributed into a pair of sets called the minimal extent and the maximal extent (Figure 10). Figure 10 presents an example of broad points, lines with vague shape and regions with broad boundary. A broad point is a zone that we approximate to a crisp region containing all elementary space portions that the point can possibly fill. The minimal extent of point is equal to its maximal extent because the shape vagueness concerns a unique topological invariant: the interior (cf. Section 3.1 for more details). For a line with vague shape (cf. Section 3.2), the minimal extent is the union of the linear parts. However, its maximal extent can contain some broad parts (i.e. presented as broad points in Figure 10(b)), at which the line can have any shape. For a region with broad boundary (cf. Section 3.3), the shape vagueness concerns the boundary. The minimal extent refers to the geometry when the boundary is as close as possible (i.e. it is drawn around the area which certainly belongs to the region). The maximal extent is the geometry of the object when the boundary is as far away as possible (i.e. it is drawn around the area, which contains all points possibly belonging to the region).

Generally, the minimal extent refers to the geometry’s parts definitely belonging to the spatial object. The maximal extent corresponds to the object’s geometry when shape vagueness is taken into account and added to the minimal extent. Outside of the maximal extent, there are no spatial points that can possibly belong to the object. The number \( n \) of crisp object types composing an object with vague shape is 1 for a broad point (i.e. a zone that we represent as a crisp region composed of the quasi-totality of possible elementary space portions that the point can fill (cf. Section 3.1)), 2 for a region with broad boundary (i.e. two crisp regions (cf. Section 3.3)) and \( n \) for lines with vague shape (i.e. 1 or \( n \) points of the line are broad (cf. Section 3.4)). For example, a region with broad boundary corresponds to a pair of crisp regions that respectively represent the minimal and maximal extents.

Each object with vague shape is composed of \( n \) crisp object types (i.e. point, line and region) distributed into a pair of sets called the minimal extent and the maximal extent (Figure 10). Figure 10 presents an example of broad points, lines with vague shape and regions with broad boundary. A broad point is a zone that we approximate to a crisp region containing all elementary space portions that the point can possibly fill. The minimal extent of point is equal to its maximal extent because the shape vagueness concerns a unique topological invariant: the interior (cf. Section 3.1 for more details). For a line with vague shape (cf. Section 3.2), the minimal extent is the union of the linear parts. However, its maximal extent can contain some broad parts (i.e. presented as broad points in Figure 10(b)), at which the line can have any shape. For a region with broad boundary (cf. Section 3.3), the shape vagueness concerns the boundary. The minimal extent refers to the geometry when the boundary is as close as possible (i.e. it is drawn around the area which certainly belongs to the region). The maximal extent is the geometry of the object when the boundary is as far away as possible (i.e. it is drawn around the area, which contains all points possibly belonging to the region).

Generally, the minimal extent refers to the geometry’s parts definitely belonging to the spatial object. The maximal extent corresponds to the object’s geometry when shape vagueness is taken into account and added to the minimal extent. Outside of the maximal extent, there are no spatial points that can possibly belong to the object. The number \( n \) of crisp object types composing an object with vague shape is 1 for a broad point (i.e. a zone that we represent as a crisp region composed of the quasi-totality of possible elementary space portions that the point can fill (cf. Section 3.1)), 2 for a region with broad boundary (i.e. two crisp regions (cf. Section 3.3)) and \( n \) for lines with vague shape (i.e. 1 or \( n \) points of the line are broad (cf. Section 3.4)). For example, a region with broad boundary corresponds to a pair of crisp regions that respectively represent the minimal and maximal extents.

Each object with vague shape is composed of \( n \) crisp object types (i.e. point, line and region) distributed into a pair of sets called the minimal extent and the maximal extent (Figure 10). Figure 10 presents an example of broad points, lines with vague shape and regions with broad boundary. A broad point is a zone that we approximate to a crisp region containing all elementary space portions that the point can possibly fill. The minimal extent of point is equal to its maximal extent because the shape vagueness concerns a unique topological invariant: the interior (cf. Section 3.1 for more details). For a line with vague shape (cf. Section 3.2), the minimal extent is the union of the linear parts. However, its maximal extent can contain some broad parts (i.e. presented as broad points in Figure 10(b)), at which the line can have any shape. For a region with broad boundary (cf. Section 3.3), the shape vagueness concerns the boundary. The minimal extent refers to the geometry when the boundary is as close as possible (i.e. it is drawn around the area which certainly belongs to the region). The maximal extent is the geometry of the object when the boundary is as far away as possible (i.e. it is drawn around the area, which contains all points possibly belonging to the region).

Generally, the minimal extent refers to the geometry’s parts definitely belonging to the spatial object. The maximal extent corresponds to the object’s geometry when shape vagueness is taken into account and added to the minimal extent. Outside of the maximal extent, there are no spatial points that can possibly belong to the object. The number \( n \) of crisp object types composing an object with vague shape is 1 for a broad point (i.e. a zone that we represent as a crisp region composed of the quasi-totality of possible elementary space portions that the point can fill (cf. Section 3.1)), 2 for a region with broad boundary (i.e. two crisp regions (cf. Section 3.3)) and \( n \) for lines with vague shape (i.e. 1 or \( n \) points of the line are broad (cf. Section 3.4)). For example, a region with broad boundary corresponds to a pair of crisp regions that respectively represent the minimal and maximal extents. This
general definition of spatial objects with vague shape is based on the following principles:

1. A spatial object with vague shape is a generalization of a crisp spatial object.
2. The minimal and the maximal extents are made up of crisp spatial object types. Only the combination of two extents corresponds to the object with vague shape.
3. For the minimal and the maximal extents, the topological invariants should be mutually disjoint.

The first principle means that the spatial extent of an object with vague shape is crisp when its minimal extent is equal to its maximal one. The second principle requires that the minimal and maximal extents verify the topological consistency conditions of the crisp spatial object types (e.g. a simple crisp region should be connected). Finally, the third principle permits the identification of topological relations based on the intersections between the topological invariants of the minimal and maximal extents of spatial objects with vague shape involved. In the next sections, we present our definitions of broad points, lines with vague shape, and regions with broad boundary.

3.1 Broad point

In the crisp context, a point \( p \) is a 0-dimensional object type which corresponds to an elementary portion of the space. This portion refers to the interior of the point (i.e. the only topological invariant of the point). Because a point does not have a boundary (the dimension of the boundary of an object with a dimension \( n \) is \( n-1 \)), the shape vagueness can characterize only the interior and thus the point itself. Semantically, a broad point occurs when an intrinsic property of the point or a lack of knowledge does not permit us to sharply distinguish the point from its neighborhood. For such a case, the spatial extent of the object is typically replaced by a zone that we represent as a crisp region composed of the family of elementary space portions that the point can fill (see an example of broad point in Figure 11). The closure\(^2\) of this crisp region represents an infinity of possible elementary space portions for the point. Consequently, a broad point does not have a minimal extent; it only has a maximal extent.

Since a simple broad point corresponds, in fact, to a simple crisp region, it should verify the following conditions:

1. The closure is a non-empty connected regular closed set.
2. The interior is a non-empty connected regular open set.
3. The boundary and exterior are connected.

To provide an example of a broad point, consider an application to help the fire brigades in their interventions. Generally, a fire fighter cannot precisely localize the fire source. However, he can draw an area in which the fire source should exist. This intervention area corresponds to a broad point and can be represented through our

<table>
<thead>
<tr>
<th>Simple Crisp point</th>
<th>Broad point</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Simple Crisp point" /></td>
<td><img src="image" alt="Broad point" /></td>
</tr>
</tbody>
</table>

Figure 11. Broad point.
model. It is clear that the size of the region representing the broad point depends on the shape vagueness level (i.e. a larger region refers to a fuzzier point).

3.2 Line with vague shape

A simple crisp line is a one-dimensional object type made up of an interior limited by two endpoints. The boundary of a crisp line corresponds to its endpoints, whereas the interior is the set of points connecting them. The shape vagueness can characterize any point of the line. Consequently, the line boundary can be partially or completely broad while the interior remains well-defined; we then speak about lines with broad boundary. In the same way, the interior can be partially or completely broad while the endpoints are well-defined; we then speak about lines with partially and completely broad interior, respectively (Figure 12). The extreme case of line shape vagueness arises when all topological invariants of the line (i.e. the interior and the boundary) are broad (Figure 12). Thus, a completely broad line arises when there is a difficulty to sharply distinguish each one of the line’s point from its neighborhood. However, a completely crisp line is a particular case of lines with vague shape, for which all of the interior and boundary are well-defined. The different cases of line shape vagueness can be combined as presented in Figure 12. To underline the different levels of line shape vagueness, we use four adverbs: (1) weakly, (2) fairly, (3) strongly and (4) completely. The term ‘weakly’ indicates that one of the topological invariants is partially broad. The term ‘fairly’ reflects either complete shape vagueness of one of topological invariants or the case where the interior and endpoints are partially broad at the same time. The term ‘strongly’ specifies complete shape vagueness for one of the topological invariants and partial shape vagueness for the second one. Finally, the term ‘completely’ is used to express total shape vagueness of the line’s components. Figure 12 shows a symmetrical matrix, in which the shape vagueness increases from ‘none’ in the upper-left cell to ‘completely’ in the lower-right cell through a progression including ‘weakly’, ‘fairly’ and ‘strongly’.

In our model, a line with vague shape is typically composed of two-dimensional parts that correspond to the line’s broad points and one-dimensional parts that represent the minimal extent of the line. Thus, the maximal extent is the union of the one- and two-dimensional parts. The interior of maximal extent corresponds to the union of the interiors of the one- and two-dimensional parts. In the same way, the boundary of the maximal extent is the union of the boundary of the one- and two-dimensional parts. Figure 13 presents the different cases of decompositions of a line with vague shape. We should note that the different representations of a line, in Figure 13, indicate a set of pictograms that we use to show different types of shape

<table>
<thead>
<tr>
<th>Crisp boundary</th>
<th>Crisp interior</th>
<th>Partially broad interior</th>
<th>Completely broad interior</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>none</td>
<td>weakly vague shape</td>
<td>fairly vague shape</td>
</tr>
<tr>
<td>Partially broad boundary</td>
<td>weakly vague shape</td>
<td>fairly vague shape</td>
<td>strongly vague shape</td>
</tr>
<tr>
<td>Completely broad boundary</td>
<td>fairly vague shape</td>
<td>strongly vague shape</td>
<td>completely vague shape</td>
</tr>
</tbody>
</table>

Figure 12. Lines with vague shape.
<table>
<thead>
<tr>
<th>Line with vague shape</th>
<th>Extents</th>
<th>Topological invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimal extent</td>
<td>-</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Boundary</td>
</tr>
<tr>
<td>Maximal extent</td>
<td>-</td>
<td>Boundary</td>
</tr>
<tr>
<td>Minimal extent</td>
<td>-</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Boundary</td>
</tr>
<tr>
<td>Maximal extent</td>
<td>-</td>
<td>Boundary</td>
</tr>
<tr>
<td>Minimal extent</td>
<td>-</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Boundary</td>
</tr>
<tr>
<td>Maximal extent</td>
<td>-</td>
<td>Boundary</td>
</tr>
<tr>
<td>Minimal extent</td>
<td>-</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Boundary</td>
</tr>
<tr>
<td>Maximal extent</td>
<td>-</td>
<td>Boundary</td>
</tr>
<tr>
<td>Minimal extent</td>
<td>-</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Boundary</td>
</tr>
<tr>
<td>Maximal extent</td>
<td>-</td>
<td>Boundary</td>
</tr>
<tr>
<td>Minimal extent</td>
<td>-</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Boundary</td>
</tr>
<tr>
<td>Maximal extent</td>
<td>-</td>
<td>Boundary</td>
</tr>
<tr>
<td>Minimal extent</td>
<td>-</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Boundary</td>
</tr>
<tr>
<td>Maximal extent</td>
<td>-</td>
<td>Boundary</td>
</tr>
<tr>
<td>Minimal extent</td>
<td>-</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Boundary</td>
</tr>
<tr>
<td>Maximal extent</td>
<td>-</td>
<td>Boundary</td>
</tr>
<tr>
<td>Minimal extent</td>
<td>-</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Boundary</td>
</tr>
<tr>
<td>Maximal extent</td>
<td>-</td>
<td>Boundary</td>
</tr>
</tbody>
</table>

---

The line does not belong to the interior

\[ \text{A broad point} \]

\[ \text{A Crisp endpoint} \]

Figure 13. Topological invariants according to the type and level of the line shape vagueness.
vagueness for lines. In other words, these representations are not based on a mathematical model that allows considering error component of spatial data as in the paper of Chrisman (1991).

More formally, a line with vague shape $\mathcal{L}$ is the union of a maximal extent $\mathcal{L}_{\text{max}}$ and a minimal extent $\mathcal{L}_{\text{min}}$. In our approach, we look for an expressive model in terms of specification of the topological relations. Consequently, we focus on the definition of the topological invariants for the maximal $\mathcal{L}_{\text{max}}$ and the minimal $\mathcal{L}_{\text{min}}$ extents. For each one, we distinguish an interior and a boundary that can be empty according to the configuration of the line (Figure 13). From a point-set topology view point, a simple line with vague shape should verify the following conditions:

1. Each one-dimensional part of a simple line with vague shape is connected.
2. Each one-dimensional part of a simple line with vague shape is not self-intersecting.
3. Each one-dimensional part of a simple line with vague shape does not form a loop.
4. Each two-dimensional part verifies the conditions defined for a broad point (Section 3.1).
5. If the endpoints are broad, they do not overlap with each other.

The three first conditions are those defined for a crisp line in the point-set topology. Then, we apply these conditions to each linear part of the line with vague shape. The fourth condition requires that each broad point on the interior or the endpoints of a line respect the topological conditions of a crisp region applied to a broad point (Section 3.1). The last condition is defined to eliminate any risk of self-intersection or loop configurations. Figure 14 shows some cases of lines that are invalid according to our model.

In the next cases, we give two examples of lines with vague shape:

- A line with weakly vague shape (e.g. with only one broad endpoint):

The Bermuda triangle is a region in the Atlantic Ocean where some aircrafts and surface vessels have disappeared. Flight 19 is the designation of five American fighters which disappeared in this triangle on December 9, 1945 (Wikipedia 2008). The five fighters left Naval Air Station of Lauderdale for a patrol. Their plan was to fly over the southeast coast before to land in Florida. However, communication was interrupted when they entered into the Bermuda Triangle. Then, only the start point (i.e. Naval Air Station of Lauderdale) and a part of the trajectory are well-known before communication interruption. The final endpoint is broad because the...
trajectory can have any shape inside the triangle. This situation can be modeled through a line with weakly vague shape.

- A line with weakly vague shape (e.g. with two crisp endpoints): ....

We suppose that an aircraft disappeared for some time from radar screens because it traversed a turbulence area. After that, the communication returns to normal and the engine arrives at its destination. In this case, the aircraft trajectory is composed of two crisp endpoints. However, the interior is partially broad, because the trajectory can take any unpredictable shape inside the turbulence area. The trajectory of the aircraft can be represented as a line with weakly vague shape.

3.3 Region with broad boundary

A crisp region is a two-dimensional spatial object type in which the shape is typically composed of an interior, a boundary and an exterior. For a region, shape vagueness occurs when there is difficulty in precisely distinguishing between the interior and the exterior via a sharp boundary. From this perspective, shape vagueness is generally correlated with the boundary, which can itself be sharp, partially broad or completely broad. It is possible to draw a minimal spatial extent by considering the boundary to be as close as possible (i.e. it is drawn around the area which certainly belongs to the region). In the same way, a maximal spatial extent can be drawn by considering the boundary to be as far as possible (i.e. it is drawn around the area which contains all of points possibly belonging to the region). Figure 15 represents an example of a region with partially broad boundary. The spatial extent of a region with broad boundary is composed of a portion called the minimal extent (i.e. all of the points definitely belonging to the spatial object) and covered by a maximal extent (i.e. all of the points possibly belonging to the spatial object).

We consider that a simple region with broad boundary is made up of two crisp regions: (1) the maximal extent, which can be ‘Equal’, ‘Contains’ or ‘Covers’; (2) the minimal extent (see examples in Figure 16). When the boundary is completely sharp (i.e. it does not contain any broad point), the region is completely crisp. This is a particular case of regions with broad boundary for which the maximal extent is equal to the minimal extent; we speak about regions with no broad boundary (or crisp regions). In the second case, the region’s boundary is broad only in some locations.

We speak about regions with partially broad boundary, where the maximal extent covers the minimal extent. For example, a forest stand or a lake can have sharp boundaries (e.g. rocky borders for a lake and a total cut for a forest stand) and broad boundaries (e.g. swamp borders for a lake) at the same time. The third case represents a typical region with broad boundary for which the boundary is completely broad. For example, the boundary of a pollution zone is broad everywhere since the pollution decreases from its kernel to the region’s exterior. In Figure 16, we present an example of each of these three cases.

![Minimal Extent](image1.png) ![Maximal Extent](image2.png)

Figure 15. Region with partially broad boundary.
Since the minimal and maximal extents are crisp regions, we distinguish three mutually disjoint topological invariants for each of them: an interior, a boundary and an exterior. Thus, a region with broad boundary $A$ is made up of six topological invariants: the interior of the minimal extent $A_{\text{min}}$, the boundary of the minimal extent $\partial A_{\text{min}}$, the exterior of the minimal extent $\neg A_{\text{min}}$, the interior of the maximal extent $A_{\text{max}}$, the boundary of the maximal extent $\partial A_{\text{max}}$ and the exterior of the maximal extent $\neg A_{\text{max}}$ (Figure 16).

- **Definition 1**: A simple region with broad boundary $A$ is composed of two simple crisp regions $A_{\text{max}}$ and $A_{\text{min}}$, where $\text{Equal}(A_{\text{max}}, A_{\text{min}})$, $\text{Contains}(A_{\text{max}}, A_{\text{min}})$ or $\text{Covers}(A_{\text{max}}, A_{\text{min}})$. $A_{\text{min}}$ is the minimal extent of $A$, $\partial A_{\text{min}}$ is the inner boundary of $A$, $A_{\text{max}}$ is the maximal extent of $A$ and $\partial A_{\text{max}}$ is the outer boundary of $A$. $A_{\text{min}}$ is the set of points certainly belonging to $A$. However, the maximal extent $A_{\text{max}}$ is the union of the minimal extent and the set of points possibly belonging to the region with broad boundary.

The following conditions should be respected for any type of regions with broad boundary:

1. The closures of the maximal and the minimal extents are non-empty regular connected closed subsets.
2. The interiors of the maximal and minimal extents are non-empty regular open sets.
3. The boundaries and exteriors of the maximal and minimal extents are connected.

<table>
<thead>
<tr>
<th>Region with broad boundary</th>
<th>Representation</th>
<th>Maximal and minimal extents</th>
<th>Topological invariants of minimal and maximal extents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Region with none broad boundary (i.e., crisp region)</td>
<td><img src="image1" alt="Diagram" /></td>
<td>Minimal extent</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Maximal extent</td>
<td>Boundary</td>
</tr>
<tr>
<td>Region with partially broad boundary (i.e., region with partially vague shape)</td>
<td><img src="image2" alt="Diagram" /></td>
<td>Minimal extent</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Maximal extent</td>
<td>Boundary</td>
</tr>
<tr>
<td>Region with completely broad boundary (i.e., region with completely vague shape)</td>
<td><img src="image3" alt="Diagram" /></td>
<td>Minimal extent</td>
<td>Interior</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Maximal extent</td>
<td>Boundary</td>
</tr>
</tbody>
</table>

Figure 16. Regions with broad boundary.

In this paper, we limit our investigations to simple regions with broad boundary (i.e., we do not consider vague regions with complex shape such as regions with broad boundary and holes or regions with broad boundary and several cores). We adopt this strategy in order to clearly present the bases of our model before improving it. Figure 17 presents some examples of invalid regions with broad boundary. In case (a), the region is invalid because its closure is non-regular, i.e. there is an isolated line that belongs to the closure. In case (b), the interior of the region is non-connected because
it is composed of two disjoint minimal extents (or cores). Then, this shape cannot be
considered as a simple region with broad boundary and therefore, it is invalid
according to our model. In case (c), the exterior does not respect the connectedness
condition of the exterior (see condition 3 presented above) since the interior contains a
hole. This type of region is considered as invalid because we only deal with simple
regions with broad boundary and without holes.

This general definition covers the crisp regions occurring when
\( \text{Equal}(\tilde{A}_{\text{max}}, \tilde{A}_{\text{min}}) \). Accordingly, this property can be used to represent a region
with only one extent and without a full membership to the object (i.e. a region
without any core; shape vagueness is about all of the region and not only about its
boundary). Our model is capable of representing this type of region but we do not
study them in detail in the present paper. Hereafter, we only focus on the typical
regions with broad boundary where \( \text{Contains}(\tilde{A}_{\text{max}}, \tilde{A}_{\text{min}}) \) or \( \text{Covers}(\tilde{A}_{\text{max}}, \tilde{A}_{\text{min}}) \)
and their topological relations.

4. Topological relations between spatial objects with vague shape

4.1 Principles

To identify the topological relations between two objects with vague shape, we
interpret their maximal and minimal extents as independent crisp geometries. In
fact, our methodology consists of identifying four specific topological relations
between the minimal and maximal extents of the objects with vague shape involved.
For that, we define a 4-Intersection matrix containing the following four topological
subrelations: \( R_1(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) \), \( R_2(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \), \( R_3(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) \), and \( R_4(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) \)
(see example in Figure 18). These topological subrelations assigned to the matrix’s
cells are those defined in the 9-Intersection model (Egenhofer and Herring 1990).
For example, if we study the topological relations between two regions with broad
boundary, each cell receives one of the eight possible topological relations between
two simple crisp regions (i.e. \text{Disjoint}, \text{Overlap}, \text{Meet}, \text{Equal}, \text{Contains}, \text{Inside},
\text{Covers}, \text{Covered by}). Then, the 4-Intersection matrix corresponds to the following
representation

\[
\begin{bmatrix}
   \tilde{B}_{\text{min}} & \tilde{B}_{\text{max}} \\
   \tilde{A}_{\text{min}} & \begin{bmatrix} R_1(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}), & R_2(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \\ R_3(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}), & R_4(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) \end{bmatrix}
\end{bmatrix}
\]

Figure 18 shows the content of the matrix that describes the topological relation
between a \textit{region with partially broad boundary} \( \tilde{A} \) and a \textit{region with completely broad
boundary} \( \tilde{B} \). In the matrix (b), the letters \( O \) and \( C \) are used to denote the relations
\text{Overlap} and \text{Contains}, respectively.
The content of the matrix corresponds to the topological subrelations relating the minimal and maximal extents of the objects involved.

Figure 18. Description of the topological relation between two regions with broad boundary: (a) visual content of the matrix; (b) formal identification of the relations between the minimal and maximal extents of the objects involved.

The content of the matrix corresponds to the topological subrelations relating the minimal and maximal extents. We use the topological subrelation between the maximal extents \( R(A_{\text{max}}, B_{\text{max}}) \) to label the global topological relation. For

<table>
<thead>
<tr>
<th>Spatial representation</th>
<th>Correspondent matrix</th>
</tr>
</thead>
</table>
| Global topological relation: Overlap | \[
\tilde{A}_{\text{min}} \ominus (\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}), \quad \tilde{A}_{\text{min}} \oplus (\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}})
\]
|                       | \[
\tilde{A}_{\text{max}} \ominus (\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}), \quad \tilde{A}_{\text{max}} \oplus (\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}})
\]
| Global topological relation: Contains | \[
\tilde{A}_{\text{min}} \ominus (\tilde{A}_{\text{min}}, \tilde{L}_{\text{min}}), \quad \tilde{A}_{\text{min}} \oplus (\tilde{A}_{\text{min}}, \tilde{L}_{\text{max}})
\]
|                       | \[
\tilde{A}_{\text{max}} \ominus (\tilde{A}_{\text{max}}, \tilde{L}_{\text{min}}), \quad \tilde{A}_{\text{max}} \oplus (\tilde{A}_{\text{max}}, \tilde{L}_{\text{max}})
\]
| Global topological relation: Overlap | \[
\tilde{L}_{\text{min}} \ominus (\tilde{L}_{\text{min}}, \tilde{L}_{\text{min}}), \quad \tilde{L}_{\text{min}} \oplus (\tilde{L}_{\text{min}}, \tilde{L}_{\text{max}})
\]
|                       | \[
\tilde{L}_{\text{max}} \ominus (\tilde{L}_{\text{max}}, \tilde{L}_{\text{min}}), \quad \tilde{L}_{\text{max}} \oplus (\tilde{L}_{\text{max}}, \tilde{L}_{\text{max}})
\]
| Global topological relation: Disjoint | \[
\tilde{P}_{\text{min}} \ominus (\tilde{P}_{\text{min}}, \tilde{P}_{\text{max}}), \quad \tilde{P}_{\text{min}} \oplus (\tilde{P}_{\text{min}}, \tilde{P}_{\text{max}})
\]
|                       | \[
\tilde{P}_{\text{max}} \ominus (\tilde{P}_{\text{max}}, \tilde{P}_{\text{min}}), \quad \tilde{P}_{\text{max}} \oplus (\tilde{P}_{\text{max}}, \tilde{P}_{\text{max}})
\]

Figure 19. Examples of identification of topological relations through a 4-Intersection matrix.
example, if $R_d(\bar{A}_{\text{max}}, \bar{B}_{\text{max}})$ is Overlap, we consider that the spatial objects with vague shape globally Overlap each other. If $R_d(\bar{A}_{\text{max}}, \bar{B}_{\text{max}})$ is Contains, we consider that the global topological relation is Contains.

In Figure 19, we present examples of the identification of topological relations between spatial objects with vague shape. The first example presents a description of the topological relation between two regions with completely broad boundary $A$ and $B$. The second example concerns a line with fairly vague shape $L$ and a region with completely broad boundary $A$. The third example shows the identification of the topological relation between two lines with fairly vague shape $L$ and $K$. Finally, the last example concerns the relation between a region with completely broad boundary $A$ and a broad point $P$.

4.2 Topological relations between a region with broad boundary and a crisp one

In our approach, the 4-Intersection matrix highlights the subrelations that exist between the components of the geometries with vague shape. Indeed, this expressivity is highlighted when the maximal extent of the spatial object with vague shape is non-empty and different from the minimal extent. In the other cases, some cells in the matrix will have the same values. For example, Figure 20 shows a region with completely broad boundary that overlaps a crisp region. The topological relation can be reduced to a 2-Intersection matrix, because the region $B$ is crisp and so its minimal extent equals its maximal one. Hereafter, we do not study topological relations that involve crisp regions. We focus on regions with different non-empty maximal extent and non-empty minimal extent.

The values assigned to the different cells of the matrix should not be arbitrarily chosen. In general, the value of $R(\bar{A}_{\text{max}}, \bar{B}_{\text{max}})$ enforces the other values. In the next section, we study these aspects specifically for the topological relations between regions with broad boundary.

4.3 Topological relations between regions with broad boundary

Eight topological relations are possible between two simple crisp regions. By considering these as the possible values in the four cells of the matrix, there are $8^4 = 4096$ possible matrices. However, definition 1 imposes a condition mandating that the extents of a region with broad boundary should be related by one of the following relations: Equal($\bar{A}_{\text{max}}, \bar{A}_{\text{min}}$), Contains($\bar{A}_{\text{max}}, \bar{A}_{\text{min}}$) or Covers($\bar{A}_{\text{max}}, \bar{A}_{\text{min}}$). Indeed, a 4-Intersection matrix cannot identify a topological relation between two regions with broad boundary when this condition is violated. Thus, the contents of the matrix’s cells are not independent. For example, if the maximal extents are disjoint, it is inconsistent for an Overlap to exist between the

Figure 20. Example of a topological relation between a region with broad boundary and a crisp region.
minimal extents (Figure 21). In Figure 21, the subrelation \( O(\tilde{A}_{\min}, \tilde{B}_{\min}) \) is gray to denote that is not allowed whereas \( D(\tilde{A}_{\min}, \tilde{B}_{\min}) \) is black to show that is permitted. Consequently, several of the 4096 possible matrices are invalid because the dependency between the matrix’s cells is not respected.

In order to enumerate the valid 4-Intersection matrices, we firstly studied the possible values in the other three cells for each of the eight possible values of \( R(\tilde{A}_{\max}, \tilde{B}_{\max}) \). For example, if \( \text{Contains}(\tilde{A}_{\max}, \tilde{B}_{\max}) \), the only possible relation between \( \tilde{A}_{\min} \) and \( \tilde{B}_{\min} \) is \( \text{Contains} \); otherwise, the expected relation cannot respect the general definition of a region with broad boundary. Figure 21 shows an example of an inconsistent matrix in which \( \text{Disjoint}(\tilde{A}_{\max}, \tilde{B}_{\max}) \) and \( \text{Contains}(\tilde{A}_{\min}, \tilde{B}_{\min}) \). This matrix is inconsistent because \( R(\tilde{B}_{\max}, \tilde{B}_{\min}) \neq \{\text{Contains, Covers, Equal}\} \). In the second step, we also fix the relation between \( \tilde{A}_{\min} \) and \( \tilde{B}_{\min} \) to deduce the possible values of \( R(\tilde{A}_{\min}, \tilde{B}_{\max}) \). For example, when \( \text{Contains}(\tilde{A}_{\max}, \tilde{B}_{\max}) \) and \( \text{Contains}(\tilde{A}_{\min}, \tilde{B}_{\min}) \), \( R(\tilde{A}_{\min}, \tilde{B}_{\max}) \) should not be \( \text{Meet or Equal} \). In this way, 31 rules (see Table 4 in the Appendix) are defined in order to ensure the consistency of matrices and to minimize the number of topological relations. In the rules’ premises, we specify either \( R(\tilde{A}_{\max}, \tilde{B}_{\max}) \) or \( (R(\tilde{A}_{\max}, \tilde{B}_{\max}) \) and \( R(\tilde{A}_{\min}, \tilde{B}_{\min}) \)). Then, we deduce the possible values in the remaining cells. In Figure 21, the matrix on the left is not valid because it requires the minimal extent to be disjoint to the minimal extent (i.e. the definition of regions with broad boundary is not respected, because \( R(\tilde{A}_{\max}, \tilde{B}_{\max}) \) should be \( \text{Contains, Covers or Equal} \)).

This study proves that only 242 topological relations are possible between two simple regions with broad boundary (see Appendix). More specifically, only one matrix is valid when \( \text{Disjoint}(\tilde{A}_{\max}, \tilde{B}_{\max}) \), 29 matrices are valid when \( \text{Contains}(\tilde{A}_{\max}, \tilde{B}_{\max}) \), 29 for \( \text{Inside}(\tilde{A}_{\max}, \tilde{B}_{\max}) \), 46 for \( \text{Covers}(\tilde{A}_{\max}, \tilde{B}_{\max}) \), 46 for \( \text{Covered by}(\tilde{A}_{\max}, \tilde{B}_{\max}) \), 65 for \( \text{Overlap}(\tilde{A}_{\max}, \tilde{B}_{\max}) \), four for \( \text{Meet}(\tilde{A}_{\max}, \tilde{B}_{\max}) \) and 22 when \( \text{Equal}(\tilde{A}_{\max}, \tilde{B}_{\max}) \). The topological relations are numbered from one to 242 according to the relation between \( \tilde{A}_{\max} \) and \( \tilde{B}_{\max} \). Table 1 shows this numbering (see Appendix to explore the relations).

5. Clustering of topological relations between regions with broad boundary

5.1 Principles

In our work, the proposed model is expressive in terms of the topological relations between regions with broad boundary. In this context, 242 topological relations are distinguished. Consequently, the clustering of relations into larger groups of relations
is an important step, because it is very difficult to keep in the mind this high number of relations. It is additionally very difficult to find a name for each one of these relations, and so the user will have difficulty of choosing the appropriate topological operator in order to express a query or integrity constraint. Mark and Egenhofer (Mark and Egenhofer 1994) studied the clustering of the topological relations between simple crisp regions and simple crisp lines both through a formal basis and by taking into account cognitive aspects. Clementini and Di Felice (Clementini and Di Felice 1997) defined a topological distance to classify the approximate topological relations between regions with completely broad boundary. In this way, they deduced 17 clusters that they represent in a conceptual neighborhood graph.

In our approach, most of the distinguished topological relations are not completely different from each other. For example, two simple regions with broad boundary can weakly or completely overlap each other depending on the content of the 4-Intersection matrix. In the first case, only the maximal extents overlap. In the second case, however, Overlap is the unique value in the matrix’s four cells. Thus, it is possible to deduce the relation’s vagueness level from the content of the 4-Intersection matrix. The objective of this section is to group the 242 topological relations into a limited number of clusters based on the content of their respective matrices.

5.2 Clustering results

In Section 4, we showed that the global topological relation is identified through a 4-Intersection matrix that enumerates four subrelations. Thus, a topological relation becomes possible if it appears at least once in the matrix. This possibility increases according to the number of similar subrelations. For example, a Covers topological relation in which Covers(\(A_{\text{max}}, B_{\text{max}}\)) and Covers(\(A_{\text{min}}, B_{\text{min}}\)) is stronger than another where only Covers(\(A_{\text{max}}, B_{\text{max}}\)). Because there are eight possible values for the matrix’s cells, we distinguish eight basic clusters that we call: DISJOINT, CONTAINS, INSIDE, COVERS, COVERED BY, EQUAL, MEET and OVERLAP. Each cluster contains all of the topological relations for which at least one of the four subrelations has the same name. For example, Figure 22 shows a topological relation that belongs to the following clusters: DISJOINT, CONTAINS, INSIDE, COVERS, COVERED BY, EQUAL, MEET and OVERLAP. Nevertheless, it belongs to the DISJOINT cluster more strongly than to the CONTAINS and COVERS clusters.

For each one of the eight basic clusters, we identify four levels of relation membership: (1) completely, (2) strongly, (3) fairly and (4) weakly (Table 2). A topological relation belongs to the cluster completely when the four subrelations are similar. It belongs to the cluster strongly when only three subrelations have the same

<table>
<thead>
<tr>
<th>The relation between ((A_{\text{max}}, B_{\text{max}}))</th>
<th>Correspondent matrices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disjoint ((A_{\text{max}}, B_{\text{max}}))</td>
<td>1</td>
</tr>
<tr>
<td>Contains ((A_{\text{max}}, B_{\text{max}}))</td>
<td>30</td>
</tr>
<tr>
<td>Equal ((A_{\text{max}}, B_{\text{max}}))</td>
<td>52</td>
</tr>
<tr>
<td>Covers ((A_{\text{max}}, B_{\text{max}}))</td>
<td>98</td>
</tr>
<tr>
<td>Covered by ((A_{\text{max}}, B_{\text{max}}))</td>
<td>144</td>
</tr>
<tr>
<td>Inside ((A_{\text{max}}, B_{\text{max}}))</td>
<td>173</td>
</tr>
<tr>
<td>Meet ((A_{\text{max}}, B_{\text{max}}))</td>
<td>177</td>
</tr>
<tr>
<td>Overlap ((A_{\text{max}}, B_{\text{max}}))</td>
<td>242</td>
</tr>
</tbody>
</table>
name as the cluster. The level labeled *fairly* contains all relations for which two subrelations have the same name as the cluster. Finally, the level called *weakly* contains the relations for which only one subrelation has the same name as the cluster. Figure 23 presents some relations that belong to different levels of *CONTAINS* and *DISJOINT* clusters, respectively, according to the contents of their correspondent matrices.

5.3 Overlapping clusters

The main result of this clustering process is a hierarchical classification of the topological relations (Figure 24). The top level is made up of eight basic clusters that each contain typically four levels: *completely*, *strongly*, *fairly* and *weakly*. The resulting 32 subclusters overlap each other because a topological relation typically belongs to different levels of one, two, three or four clusters at the same time. For example, topological relation number 56 (see Appendix and Table 2) belongs *fairly* to the *CONTAINS* cluster and *weakly* to the *COVERS* and *INSIDE* clusters. The bottom level of the classification contains the 242 topological relations that appear in different subclusters.

6. Specification of spatial queries and integrity constraints

In the previous sections, we presented a framework for identifying topological relations between regions with broad boundary. Because it uses the 9-Intersection model (Egenhofer and Herring 1990), our model can be easily integrated in a spatial database system. Indeed, the SQL language can be extended in order to retrieve regions with broad boundary based on the qualitative information given by the user regarding their topological relation. In fact, a topological relation between two regions with broad boundary can be recognized through the combination of four crisp topological operators. For example, relation no. 56 corresponds to (*Disjoint, Disjoint, Contains, Covers*). Hereafter, we suppose that we integrated our spatial model in a relational engine in order to give an example of its possible use in spatial queries involving regions with broad boundary. We suppose that the spatial database stores pollution zones, which are represented as regions with broad boundary. In the first query example, the user gives a coarse description of the topological relation when he introduces the specification *fairly DISJOINT*. The query’s results should contain the pollution zones related to zone *A* by a topological relation belonging to this subcluster. In the second example, the query is more specific because the user identifies all topological subrelations that relate (*A*<sub>min</sub>, *B*<sub>min</sub>),
Table 2. Clustering results.

<table>
<thead>
<tr>
<th>Cluster's name</th>
<th>Vagueness level</th>
<th>Topological relations' numbers (cf. Appendix)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>DISJOINT</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Weakly</td>
<td>13, 14, 15, 17, 41, 42, 43, 44, 67, 69, 70, 71, 72, 74, 75, 80, 113, 115, 116, 117, 118, 120, 121, 126, 157, 159, 161, 162, 193, 194, 195, 196, 197, 198, 199, 200, 201, 204, 205, 208, 213, 214, 215, 216</td>
</tr>
<tr>
<td></td>
<td>Fairly</td>
<td>16, 73, 76, 119, 122, 158, 175, 176, 202, 203, 206, 207, 209, 210, 211, 212</td>
</tr>
<tr>
<td></td>
<td>Strongly</td>
<td>174, 217</td>
</tr>
<tr>
<td></td>
<td>Completely</td>
<td>1</td>
</tr>
<tr>
<td><strong>CONTAINS</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fairly</td>
<td>8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 32, 33, 37, 53, 54, 55, 56, 60, 103, 104, 152, 178, 179, 180</td>
</tr>
<tr>
<td></td>
<td>Strongly</td>
<td>2, 3, 4, 5, 7</td>
</tr>
<tr>
<td></td>
<td>Completely</td>
<td>6</td>
</tr>
<tr>
<td><strong>EQUAL</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fairly</td>
<td>24, 30, 31, 169</td>
</tr>
<tr>
<td></td>
<td>Strongly</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Completely</td>
<td></td>
</tr>
<tr>
<td><strong>COVERS</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fairly</td>
<td>20, 49, 54, 58, 62, 64, 65, 66, 70, 72, 74, 75, 78, 82, 85, 86, 89, 90, 91, 92, 97, 98, 136, 142, 168, 220, 222</td>
</tr>
<tr>
<td></td>
<td>Strongly</td>
<td>83, 84, 87</td>
</tr>
<tr>
<td></td>
<td>Completely</td>
<td>81</td>
</tr>
<tr>
<td><strong>COVERED BY</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Strongly</td>
<td>129, 131, 134</td>
</tr>
<tr>
<td></td>
<td>Completely</td>
<td>127</td>
</tr>
</tbody>
</table>
Table 2. (Continued.)

<table>
<thead>
<tr>
<th>Cluster’s name</th>
<th>Vagueness level</th>
<th>Topological relations’ numbers (cf. Appendix)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Strongly</td>
<td>146, 147, 148, 149, 151</td>
</tr>
<tr>
<td></td>
<td>Completely</td>
<td>150</td>
</tr>
<tr>
<td>MEET</td>
<td>Weakly</td>
<td>9, 11, 12, 15, 36, 38, 39, 40, 59, 61, 62, 63, 64, 66, 74, 80, 105, 107, 108, 109, 110, 112, 120, 126, 153, 155, 156, 161, 174, 184, 185, 186, 187, 188, 189, 190, 191, 192, 204, 205, 206, 207, 213, 214, 215, 216</td>
</tr>
<tr>
<td></td>
<td>Fairly</td>
<td>10, 65, 68, 111, 114, 154, 175, 176, 208</td>
</tr>
<tr>
<td></td>
<td>Strongly</td>
<td>177</td>
</tr>
<tr>
<td></td>
<td>Completely</td>
<td>177</td>
</tr>
<tr>
<td></td>
<td>Fairly</td>
<td>27, 96, 97, 143, 144, 172, 179, 182, 188, 189, 191, 192, 197, 198, 200, 201, 202, 203, 204, 205, 221, 222, 224, 229, 234, 235, 240, 241</td>
</tr>
<tr>
<td></td>
<td>Strongly</td>
<td>190, 199, 236, 238, 239, 242</td>
</tr>
<tr>
<td></td>
<td>Completely</td>
<td>237</td>
</tr>
</tbody>
</table>

\((A_{\min}, B_{\max}), (A_{\max}, B_{\min})\) and \((A_{\max}, B_{\max})\). The third example shows another use of our model, in which it is possible to display the different strength levels of a relation (e.g. weakly Overlap or strongly Overlap) that occurs between two regions with broad

![Figure 23. Evaluation of a topological relation’s membership to one of the eight basic clusters.](image-url)
boundary (cf. Table 3). Table 3 shows a possible result for the query presented in example 3.

**Example 1:**

```sql
SELECT Pollution_Zone.geometry FROM Pollution_Zone
WHERE vague_Relate(pollution_zone.geometry, A.geometry, fairly DISJOINT);
```

**Example 2:**

```sql
SELECT Pollution_Zone.geometry FROM Pollution_Zone
WHERE vague_Relate(Pollution_Zone.geometry, A.geometry, Disjoint, Meet, Contains, Contains);
```

**Example 3:**

```sql
SELECT P1.id, P2.id, determine(P1.geometry, P2.geometry, 'Overlap')
FROM Pollution_Zone P1, P2
WHERE P1.id.<>P2.id;
```

In the same way, it is possible to use the model to formally express spatial integrity constraints for objects with vague shape. For example, let the constraint saying that ‘two different lakes can be only fairly meet or completely disjoint’. This constraint can be formally expressed by integrating new spatial operators (e.g. completely Contains, weakly Covers, etc.) in a formal constraint language like the OCL (Pinet et al. 2007). The database storing the lakes is consistent only if the topological relations between the different entities belong to fairly MEET or completely DISJOINT subclusters (see example 4).

**Example 4:**

```sql
Context Lake inv:

Lake.allInstances->forAll (a, b | a<>b implies fairly MEET(a,b) or completely DISJOINT(a,b));
```

<table>
<thead>
<tr>
<th>P1.id</th>
<th>P2.id</th>
<th>Determine</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>23</td>
<td>Weakly overlap</td>
</tr>
<tr>
<td>45</td>
<td>14</td>
<td>-</td>
</tr>
<tr>
<td>18</td>
<td>26</td>
<td>Strongly overlap</td>
</tr>
</tbody>
</table>
7. Discussion

Clementini and Di Felice (Clementini and Di Felice 1997) propose an extension of the 9-Intersection model (Egenhofer and Herring 1990) that uses a broad boundary to replace the sharp boundary. In this approach, 44 topological relations are distinguished between two regions with broad boundary. By considering a topological distance, Clementini and Di Felice (Clementini and Di Felice 1997) draw a conceptual neighborhood graph that shows similarity degrees between relations classified into 17 clusters. The main advantage of this approach is the ability to support a coarser spatial reasoning involving regions with broad boundary. When the needs are more specific, it becomes more difficult to use this model. Furthermore, the identification of a broad boundary as a two-dimensional topological invariant requires respecting consistency conditions related to closeness and connectedness. Tang (Tang 2004) presents a more expressive model than that of (Clementini and Di Felice 1997), because he decomposes the broad boundary into the boundary's interior and the boundary's boundary. Based on this definition, Tang (Tang 2004) presents another extension of 9-Intersection model, in which topological relations are identified through a $4 \times 4$-Intersection matrix. He distinguishes 152 topological relations presented as variants of the 44 relations from (Clementini and Di Felice 1997). Nonetheless, this model does not distinguish between the boundaries of the minimal and maximal extents. Accordingly, many topological relations cannot be distinguished (see examples in Section 2.3 and Section 7). Moreover, regions with partially broad boundary (see example in Figure 2) are considered invalid and cannot be presented through existing exact models. In our approach, we resolve this problem by considering a simple region with broad boundary as a general concept which can be specialized into: regions with none broad boundary (or crisp regions), regions with partially broad boundary and regions with completely broad boundary. A region is then defined as a maximal extent and a minimal extent, in which either \( \text{Equal}(\mathcal{A}_{\text{max}}, \mathcal{A}_{\text{min}}) \) or \( \text{Contains}(\mathcal{A}_{\text{max}}, \mathcal{A}_{\text{min}}) \) or \( \text{Covers}(\mathcal{A}_{\text{max}}, \mathcal{A}_{\text{min}}) \). The notion of broad boundary (i.e. in the sense of connected and closed polygonal zone) is not formally defined as a topological invariant in our model. It can be deduced from the difference between the minimal extent and the maximal one. This difference can be non-empty everywhere around the minimal extent (i.e. region with completely broad boundary), non-empty in some location and empty in some others (i.e. region with partially broad boundary) or empty everywhere around the minimal extent (i.e. crisp region). Our main motivations for adopting this framework are to consider regions with partially broad boundary and to present an expressive model in terms of the identification of topological relations between regions with broad boundary. With regard to principal exact models (Clementini and Di Felice 1997; Cohn and Gotts 1996; Erwig and Schneider 1997; Tang 2004), our approach allows making distinction between partial shape vagueness and complete shape vagueness. This distinction is very important in order to deal with two main problems: an ontological problem and a modeling one. First, the ontological problem means that ‘shape vagueness’ should not be considered as a ‘binary imperfection’ (i.e. only two possibilities are considered for an object’s shape: crisp or vague). Spatial objects can be characterized by different levels of shape vagueness (e.g. how can we classify a region with partially broad boundary? Is it a crisp or a vague region?). These levels are easily computed in fuzzy models by using a quantitative approach. In our submission, we try to categorize two levels using a qualitative approach because we believe that ‘shape vagueness’ is a qualitative
problem. It is clear that our approach cannot provide a fine computation of shape vagueness as in fuzzy models. However, we believe that our model provides a solution to qualitatively distinguish different levels of shape vagueness in the category of exact models. We do not claim that exact models are better than fuzzy ones, because the needs are not identical and therefore, the direct comparison is not appropriate. Second, the modeling problem refers to the lack of expressivity in existing exact models to represent the objects, which include sharpness and broadness in their topological invariants at the same time. To deal with this second problem, our model can formally represent regions with partially broad boundary in addition to those with completely broad boundary. This distinction is ignored in the most existing exact models, notably Clementini and Di Felice (1997), Cohn and Gotts (1996), Erwig and Schneider (1997) and Tang (2004). For lines, nine types of lines with vague shape are distinguished by considering four levels of shape vagueness: weakly, fairly, strongly and completely. For topological relations, we propose a 4-Intersection matrix where it is possible to identify respective sub-relations between minimal extents and maximal ones: \((A_{\text{min}}, B_{\text{min}}), (A_{\text{min}}, B_{\text{max}}), (A_{\text{max}}, B_{\text{min}})\) and \((A_{\text{max}}, B_{\text{max}})\). These subrelations are labeled using the 9-Intersection model (Egenhofer and Herring 1990). In our paper, 31 rules (or strategies) have been defined in order to minimize the number of topological relations between regions with broad boundary and to control their consistency. In this context, we would clarify that the seven first strategies defined in Schmitz and Morris (2006) can be considered as a subset of our 31 rules (see these rules in the appendix). More specifically, Strategy 1 (Schmitz and Morris 2006) can correspond to Rule 1 in our model, Strategy 2\(\iff\)Rule 2, Strategy 3\(\iff\)Rule 3, Strategy 4\(\iff\)Rule 3 (this rule is applied for Inside and Contains relations), Strategy 5\(\iff\)Rule 5, Strategy 6\(\iff\)Rule 6 and Strategy 7\(\iff\)Rule 6 (this rule is applied for Inside and Contains relations). The eighth strategy presented in the paper of Schmitz and Morris (2006) does not provide any indication about the appropriate topological subrelations when overlap relations arise between components of regions with broad boundary involved (i.e. it recommends additional investigations). However, in our paper, we propose eight strategies when an Overlap relation occurs between maximal extents of two regions with broad boundary (Rule 20–Rule 27). Then, incoherent and redundant topological relations have been removed using the 31 rules presented in the Appendix. We distinguish 242 different topological relations that we classify into eight overlapping basic clusters. Each cluster has four membership levels (or subclusters): completely, strongly, fairly and weakly. This classification of the topological relations is proposed to support an adverbial expression of topological constraints. Nevertheless, our model is not able to quantify the gradual change inside the maximal extent in the same way that the fuzzy approaches do (Zhan 1997; Schneider 2001; Du et al. 2005; Dilo 2006; Verstraete et al. 2007). Finally, we are convinced that a more detailed comparison of the models’ expressivity requires to be thoughtfully investigated in another paper.

The Egg–Yolk model (Cohn and Gotts 1996) was our main inspiration to develop this framework for identifying topological relations. However, there are some fundamental differences between our model and that of Cohn and Gotts (1996). For instance, the topological relations used in Cohn and Gotts (1996) are those defined in the RCC-5 model (Randell and Cohn 1989; Cohn et al. 1997). In contrast, the topological relations used in the cells of our matrix are those defined in the 9-Intersection model (Egenhofer and Herring 1990). It is true that we follow
the same methodology to identify topological relations. However, our definitions of objects with vague shape are substantially different. Our model is based on the point-set theory where points and lines are considered as additional basic crisp spatial object types. In terms of originality, we do not formally redefine the concept ‘broad boundary’ as it is carried out in most of existing exact models. Our approach is based on the distinction between a minimal extent and a maximal one. The broad boundary can be deduced from the difference between these two extents, but it is not defined as a topological invariant of the object. In the paper of Cohn and Gotts (1996), a conceptual neighborhood graph was drawn with 44 topological relations which are classified into 13 clusters. In our model, we define a hierarchical classification based on the content of the matrices we use to identify the topological relations. This classification is the basis of an adverbial approach that we use to specify topological constraints between regions with broad boundary.

8. Conclusions and future work

Shape vagueness is an inherent property of many spatial objects like lakes, valleys and mountains. In GIS and spatial databases, it is a general practice to neglect shape vagueness and formally represent spatial objects with vague shape as crisp geometries. Using such inappropriate representations can provide a source of spatial data quality degradation, because the reliability of spatial data is decreased. With emergence of prediction applications, data integration and strategic decisional needs, researchers are increasingly more motivated to propose different methods for the formal representation of shape vagueness. A review of the literature regarding this topic proves that existing exact models do not permit the representation of objects with partially vague shape. For such objects, shape vagueness partially characterizes one or several of its topological invariants. For example, a lake can have rocky banks on one side and swamp banks on the other side at the same time; the boundary is broad only for the swamp part. In this work, we have proposed an exact model in order to represent spatial objects that can have: crisp shapes, partially vague shapes or completely vague shapes. We have considered this categorization of shape vagueness during the identification of topological relations.

More specifically, this paper contributes in three main ways. Based on point-set topology, we firstly define three basic types of spatial objects with vague shape: broad point, line with vague shape (i.e. lines with broad boundary, lines with broad interior or broad lines) and region with broad boundary. Each one of them is typically defined as a minimal extent \(A_{\min}\) and a maximal extent \(A_{\max}\), and these extents must verify some topological conditions in order to be valid. This model permits the representation of spatial objects with partially vague shape considered as invalid in the existing models of Clementini and Di Felice (1997); Tang (2004); Reis et al. (2006). Then, we identify a topological relation through use of a 4-Intersection matrix that permits the enumeration of four subrelations: \(R_1(A_{\min}, B_{\min})\), \(R_2(A_{\min}, B_{\max})\), \(R_3(A_{\max}, B_{\min})\) and \(R_4(A_{\max}, B_{\max})\). Using this formalism for simple regions with broad boundary, 242 relations can be distinguished (cf. Appendix). In order to retain our propositions useful in practice, we propose the clustering of these topological relations. A topological relation can belong to one or several clusters with various qualitative strengths: completely, strongly, fairly and weakly. The objective of this qualitative clustering is to improve the specification of spatial queries and integrity constraints involving spatial objects with vague shape.
In this paper, our study is limited to the regions with broad boundary which are composed by a simple core (or minimal extent). Extending this approach to regions with more complex shapes (e.g. regions with broad boundary and holes, regions with several cores, regions composed by disjoint subregions with different memberships to the regions, etc.) is one of our future researches. We are conscious that it can be a limitation of our current model but considering this type of region requires additional investigations which exceed the objectives of this paper. The goal of this paper is to clearly present the basis of our approach before improving it.

Another extension consists of using this approach to improve the logical consistency of spatial databases involving spatial objects with vague shape. More specifically, we are interested in the specification of integrity constraints in spatial databases storing objects with vague shape. We hope to identify both integrity constraint categories and the requirements for their formal expression. The framework presented earlier can provide a basis for the extension of a formal constraint language like OCL (Pinet et al. 2007) to express tolerant integrity constraints for objects with vague shape.

Finally, this approach can be used to deal with geometric heterogeneities between sources databases in decisional applications. These applications require the integration of spatial data from heterogeneous sources before they are stored in a spatial data warehouse (Bédard et al. 2007). The main difficulty lies in choosing one of the available geometric representations. We suggest merging the different representations in such a way that the result looks like a spatial object with vague shape. The tolerant integrity constraints can be used to increase the logical consistency of such data.

Notes
1. The region is defined as the elementary component of the space, i.e. points and lines are not considered.
2. The closure, in point set topology, is the union of the interior and the boundary.
3. The spatial relations (i.e. Equal, Contains, Covers) used in this definition are those defined in Egenhofer and Herring (1990).

References


SHU, H., SPACCAPIETRA, S., PARENT, C. and QUESADA SEDAS, S., 2003, Uncertainty of Geographic Information and its Support in MADS. In *Proceeding of the 2nd
International Symposium on Spatial Data Quality (Hong Kong: The Hong Kong Polytechnic University), pp. 1–13.


VAN OORT, P., 2006, Spatial Data Quality: from Description to Application, Publication on Geodesy 60 (Delft: NCG).


Appendix: 242 topological relations between regions with broad boundary and required rules to deduce them (Figure 25)

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Disjoint</td>
<td>Disjoint</td>
<td>Disjoint</td>
</tr>
<tr>
<td>2</td>
<td>Contains</td>
<td>Inside</td>
<td>Contains</td>
</tr>
<tr>
<td>3</td>
<td>Contains</td>
<td>Covered by</td>
<td>Contains</td>
</tr>
<tr>
<td>4</td>
<td>Contains</td>
<td>Equal</td>
<td>Contains</td>
</tr>
<tr>
<td>5</td>
<td>Contains</td>
<td>Overlap</td>
<td>Contains</td>
</tr>
<tr>
<td>6</td>
<td>Contains</td>
<td>Inside</td>
<td>Contains</td>
</tr>
<tr>
<td>7</td>
<td>Contains</td>
<td>Covers</td>
<td>Contains</td>
</tr>
<tr>
<td>8</td>
<td>Inside</td>
<td>Inside</td>
<td>Contains</td>
</tr>
<tr>
<td>9</td>
<td>Meet</td>
<td>Contains</td>
<td>Contains</td>
</tr>
<tr>
<td>10</td>
<td>Meet</td>
<td>Meet</td>
<td>Contains</td>
</tr>
<tr>
<td>11</td>
<td>Meet</td>
<td>Overlap</td>
<td>Contains</td>
</tr>
<tr>
<td>12</td>
<td>Meet</td>
<td>Covered by</td>
<td>Contains</td>
</tr>
<tr>
<td>13</td>
<td>Disjoint</td>
<td>Inside</td>
<td>Contains</td>
</tr>
<tr>
<td>14</td>
<td>Disjoint</td>
<td>Covered by</td>
<td>Contains</td>
</tr>
<tr>
<td>15</td>
<td>Disjoint</td>
<td>Meet</td>
<td>Contains</td>
</tr>
<tr>
<td>16</td>
<td>Disjoint</td>
<td>Disjoint</td>
<td>Contains</td>
</tr>
<tr>
<td>17</td>
<td>Disjoint</td>
<td>Overlap</td>
<td>Contains</td>
</tr>
<tr>
<td>18</td>
<td>Covers</td>
<td>Inside</td>
<td>Contains</td>
</tr>
<tr>
<td>19</td>
<td>Covers</td>
<td>Overlap</td>
<td>Contains</td>
</tr>
<tr>
<td>20</td>
<td>Covers</td>
<td>Covers</td>
<td>Contains</td>
</tr>
<tr>
<td>21</td>
<td>Covers</td>
<td>Covered by</td>
<td>Contains</td>
</tr>
<tr>
<td>22</td>
<td>Covers</td>
<td>Inside</td>
<td>Contains</td>
</tr>
<tr>
<td>23</td>
<td>Inside</td>
<td>Covered by</td>
<td>Contains</td>
</tr>
<tr>
<td>24</td>
<td>Equal</td>
<td>Equal</td>
<td>Contains</td>
</tr>
<tr>
<td>25</td>
<td>Covers</td>
<td>Equal</td>
<td>Contains</td>
</tr>
<tr>
<td>26</td>
<td>Equal</td>
<td>Covered by</td>
<td>Contains</td>
</tr>
<tr>
<td>27</td>
<td>Equal</td>
<td>Inside</td>
<td>Contains</td>
</tr>
<tr>
<td>28</td>
<td>Overlap</td>
<td>Overlap</td>
<td>Contains</td>
</tr>
<tr>
<td>29</td>
<td>Equal</td>
<td>Inside</td>
<td>Contains</td>
</tr>
<tr>
<td>30</td>
<td>Equal</td>
<td>Covers</td>
<td>Equal</td>
</tr>
<tr>
<td>31</td>
<td>Equal</td>
<td>Inside</td>
<td>Equal</td>
</tr>
<tr>
<td>32</td>
<td>Contains</td>
<td>Inside</td>
<td>Equal</td>
</tr>
<tr>
<td>33</td>
<td>Overlap</td>
<td>Covered by</td>
<td>Contains</td>
</tr>
<tr>
<td>34</td>
<td>Inside</td>
<td>Inside</td>
<td>Equal</td>
</tr>
<tr>
<td>35</td>
<td>Inside</td>
<td>Inside</td>
<td>Covers</td>
</tr>
<tr>
<td>36</td>
<td>Meet</td>
<td>Inside</td>
<td>Contains</td>
</tr>
<tr>
<td>37</td>
<td>Contains</td>
<td>Covered by</td>
<td>Equal</td>
</tr>
<tr>
<td>38</td>
<td>Meet</td>
<td>Inside</td>
<td>Equal</td>
</tr>
<tr>
<td>39</td>
<td>Meet</td>
<td>Covered by</td>
<td>Equal</td>
</tr>
<tr>
<td>40</td>
<td>Meet</td>
<td>Covered by</td>
<td>Equal</td>
</tr>
<tr>
<td>41</td>
<td>Disjoint</td>
<td>Inside</td>
<td>Covers</td>
</tr>
<tr>
<td>42</td>
<td>Disjoint</td>
<td>Covered by</td>
<td>Equal</td>
</tr>
<tr>
<td>43</td>
<td>Disjoint</td>
<td>Inside</td>
<td>Contains</td>
</tr>
<tr>
<td>44</td>
<td>Disjoint</td>
<td>Covered by</td>
<td>Contains</td>
</tr>
<tr>
<td>45</td>
<td>Overlap</td>
<td>Covered by</td>
<td>Contains</td>
</tr>
<tr>
<td>46</td>
<td>Overlap</td>
<td>Inside</td>
<td>Covers</td>
</tr>
<tr>
<td>47</td>
<td>Overlap</td>
<td>Covered by</td>
<td>Equal</td>
</tr>
<tr>
<td>48</td>
<td>Overlap</td>
<td>Inside</td>
<td>Contains</td>
</tr>
</tbody>
</table>

Figure 25. Two hundred and forty-two topological relations between regions with broad boundary.
Rules

Table 4. Required rules for topological relations between regions with broad boundary.

<table>
<thead>
<tr>
<th>Rule 1</th>
<th>Let $A$ and $B$ two simple regions with broad boundary, if $\text{Disjoint}(\tilde{A}<em>{\max}, \tilde{B}</em>{\max})$ then $\text{Disjoint}(\tilde{A}<em>{\min}, \tilde{B}</em>{\min})$.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\begin{align*} \tilde{A}<em>{\min} &amp; \in \text{D}(\tilde{A}</em>{\min}, \tilde{B}<em>{\min}) \ \tilde{A}</em>{\max} &amp; \in \text{D}(\tilde{A}<em>{\max}, \tilde{B}</em>{\max}) \end{align*}$</td>
</tr>
</tbody>
</table>

Proof: Let $A$ and $B$ two simple regions with broad boundary where $\text{Disjoint}(\tilde{A}_{\max}, \tilde{B}_{\max})$. Now, we suppose that $R(\tilde{A}_{\min}, \tilde{B}_{\min}) \neq \text{Disjoint}$. In this case, the relation between minimal extent $\tilde{A}_{\min}$ and maximal extent $\tilde{A}_{\max}$ of a region with broad boundary $A$ or that between $\tilde{B}_{\max}$ and $\tilde{B}_{\min}$ does not correspond to $\text{Contains}$, $\text{Covers}$, $\text{Equal}$. Thus, there is a contradiction with definition 1.

<table>
<thead>
<tr>
<th>Rule 2</th>
<th>Let $A$ and $B$ two regions with broad boundary, if $\text{Meet}(\tilde{A}<em>{\max}, \tilde{B}</em>{\max})$ then $\text{Meet}(\tilde{A}<em>{\min}, \tilde{B}</em>{\min})$.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\begin{align*} \tilde{A}<em>{\min} &amp; \in {D(\tilde{A}</em>{\min}, \tilde{B}<em>{\min}) } \ \tilde{A}</em>{\max} &amp; \in {M(\tilde{A}<em>{\max}, \tilde{B}</em>{\max}) } \end{align*}$</td>
</tr>
</tbody>
</table>

Proof: Let $A$ and $B$ two simple regions with broad boundary where $\text{Meet}(\tilde{A}_{\max}, \tilde{B}_{\max})$. Now, we suppose that $R(\tilde{A}_{\min}, \tilde{B}_{\min}) \notin \{\text{Disjoint, Meet}\}$. In this case, relation between minimal extent $\tilde{A}_{\min}$ and maximal extent $\tilde{A}_{\max}$ or that between $\tilde{B}_{\max}$ of a region with broad boundary $A$ or that between $\tilde{B}_{\min}$ and $\tilde{B}_{\max}$ does not correspond to $\text{Contains}$, $\text{Covers}$, $\text{Equal}$. Thus, there is a contradiction with definition 1.

<table>
<thead>
<tr>
<th>Rule 3</th>
<th>Let $A$ and $B$ two regions with broad boundary, if $\text{Contains}(\tilde{A}<em>{\max}, \tilde{B}</em>{\max})$ then $\text{Contains}(\tilde{A}<em>{\min}, \tilde{B}</em>{\min})$.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\begin{align*} \tilde{A}<em>{\min} &amp; \in C(\tilde{A}</em>{\max, \min}) \ \tilde{A}<em>{\max} &amp; \in C(\tilde{A}</em>{\max, \min}) \end{align*}$</td>
</tr>
</tbody>
</table>

Proof: Let $A$ and $B$ two simple regions with broad boundary where $\text{Contains}(\tilde{A}_{\max, \max})$. According to definition 1, any region with broad boundary $A$ should respect the principal following condition: $\text{Equal}(\tilde{A}_{\max, \min})$, $\text{Contains}(\tilde{A}_{\max, \min})$, or $\text{Covers}(\tilde{A}_{\max, \min})$. Moreover, $\text{Contains}$ is a transitive topological relation: if $\text{Contains}(A, B)$ and $\text{Contains}(B, C) \rightarrow \text{Contains}(A, C)$. Then, since $\text{Contains}(\tilde{A}_{\max, \max})$ and $R(\tilde{B}_{\max, \min}) = \{\text{Contains, Covers, Equal}\}$ then $\text{Contains}(\tilde{A}_{\max, \min})$ and vice versa.

<table>
<thead>
<tr>
<th>Rule 4</th>
<th>Let $A$ and $B$ two regions with broad boundary, if $\text{Covers}(\tilde{A}<em>{\max, \max})$ then $\text{Covers}(\tilde{A}</em>{\min, \min})$.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\begin{align*} \tilde{A}<em>{\min} &amp; \in \text{C}(\tilde{A}</em>{\max, \min}) \ \tilde{A}<em>{\max} &amp; \in \text{C}(\tilde{A}</em>{\max, \min}) \end{align*}$</td>
</tr>
</tbody>
</table>

Proof: Let $A$ and $B$ two simple regions with broad boundary where $\text{Covers}(\tilde{A}_{\max, \max})$. According to definition 1, any region with broad boundary $A$ should respect the principal following condition: $\text{Equal}(\tilde{A}_{\max, \min})$, $\text{Contains}(\tilde{A}_{\max, \min})$ or $\text{Covers}(\tilde{A}_{\max, \min})$. $\text{Contains}$ is a transitive topological relation: if $\text{Contains}(A, B)$ and $\text{Contains}(B, C) \rightarrow \text{Contains}(A, C)$. Then, if $\text{Contains}(\tilde{B}_{\max, \min})$ then $\text{Contains}(\tilde{A}_{\max, \min})$ else if $R(\tilde{B}_{\max, \min}) = \{\text{Covers, Equal}\}$ then $\text{Covers}(\tilde{A}_{\max, \min})$ else if $\text{Covers}(\tilde{B}_{\max, \min})$ then $R(\tilde{A}_{\max, \min}) = \{\text{Contains, Covers}\}$ and vice versa.
Table 4. (Continued.)

**Rule 5:** Let $\tilde{A}$ and $\tilde{B}$ two regions with broad boundary, if $\text{Equal} (\tilde{A}_\text{min}, \tilde{B}_\text{max})$ then $R (\tilde{A}_\text{max}, \tilde{B}_\text{max})$ and $\text{Contains} (\tilde{A}_\text{max}, \tilde{B}_\text{min})$.

![Formula](attachment:image.png)

**Proof:** Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary where $\text{Equal} (\tilde{A}_\text{max}, \tilde{B}_\text{max})$. According to definition 1, any region with broad boundary $\tilde{A}$ should respect the principal following condition: $\text{Equal} (\tilde{A}_\text{max}, \tilde{A}_\text{min})$. $\text{Contains} (\tilde{A}_\text{max}, \tilde{A}_\text{min})$ or $\text{Covers} (\tilde{A}_\text{max}, \tilde{A}_\text{min})$. In this case, we do not consider $\text{Equal} (\tilde{A}_\text{max}, \tilde{A}_\text{min})$ because the topological relation becomes between crisp regions thoughtfully studied in other works (e.g. Egenhofer and Herring 1990). $\text{Equal}$ and $\text{Contains}$ are transitive topological relations: $\text{Equal}(A, B)$ and $\text{Equal}(B, C) \rightarrow \text{Equal}(A, C)$, $\text{Contains}(A, B)$ and $\text{Contains}(B, C) \rightarrow \text{Contains}(A, C)$. Then, if $\text{Equal} (\tilde{A}_\text{max}, \tilde{B}_\text{max})$ and $\text{Contains} (\tilde{B}_\text{max}, \tilde{B}_\text{min})$ then $\text{Contains} (\tilde{A}_\text{max}, \tilde{B}_\text{min})$ (1) else if $\text{Covers} (\tilde{B}_\text{max}, \tilde{B}_\text{min})$ then $\text{Covers} (\tilde{A}_\text{max}, \tilde{B}_\text{min})$ (2). Then, (1) and (2) implies that $R (\tilde{A}_\text{max}, \tilde{B}_\text{min}) \in \{ \text{Contains}, \text{Covers} \}$ and $R (\tilde{B}_\text{max}, \tilde{A}_\text{min}) \in \{ \text{Inside}, \text{Covered by} \}$.

**Rule 6:** Let $\tilde{A}$ and $\tilde{B}$ two regions with broad boundary, if $\text{Contains} (\tilde{A}_\text{max}, \tilde{B}_\text{max})$ and $\text{Contains} (\tilde{A}_\text{min}, \tilde{B}_\text{min})$ then $R (\tilde{A}_\text{max}, \tilde{B}_\text{max})$ and $\text{Contains} (\tilde{A}_\text{min}, \tilde{B}_\text{min})$.

![Formula](attachment:image2.png)

**Proof:** Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary where $\text{Contains} (\tilde{A}_\text{max}, \tilde{B}_\text{max})$ and $\text{Contains} (\tilde{A}_\text{min}, \tilde{B}_\text{min})$. According to definition 1, we have $\text{Equal} (\tilde{A}_\text{max}, \tilde{A}_\text{min})$. $\text{Contains}(\tilde{A}_\text{max}, \tilde{A}_\text{min})$ or $\text{Covers}(\tilde{A}_\text{max}, \tilde{A}_\text{min})$). We suppose now that $\text{Disjoint} (\tilde{A}_\text{min}, \tilde{B}_\text{max})$ or $\text{Meet} (\tilde{A}_\text{min}, \tilde{B}_\text{max})$ (1). By considering definition 1 and $\text{Contains} (\tilde{A}_\text{max}, \tilde{B}_\text{max})$, since $R (\tilde{B}_\text{max}, \tilde{B}_\text{min}) \{ \text{Contains}, \text{Covers}, \text{Equal} \}$ then $\text{Contains} (\tilde{A}_\text{max}, \tilde{B}_\text{min})$ (2). In addition, since $\text{Contains} (\tilde{A}_\text{min}, \tilde{B}_\text{min})$ and (1) then $R (\tilde{B}_\text{max}, \tilde{B}_\text{min}) \notin \{ \text{Contains}, \text{Covers}, \text{Equal} \}$. Thus, there is a contradiction with definition 1.

**Rule 7:** Let $\tilde{A}$ and $\tilde{B}$ two regions with broad boundary, if $\text{Contains} (\tilde{A}_\text{max}, \tilde{B}_\text{max})$ and $\text{Inside}(\tilde{A}_\text{min}, \tilde{B}_\text{min})$.

![Formula](attachment:image3.png)

**Proof:** Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary where $\text{Contains} (\tilde{A}_\text{max}, \tilde{B}_\text{max})$ and $\text{Inside}(\tilde{A}_\text{min}, \tilde{B}_\text{min})$. We suppose now that $R (\tilde{A}_\text{min}, \tilde{B}_\text{max}) \notin \{ \text{Inside} \}$ (1). By considering definition 1 and $\text{Contains} (\tilde{A}_\text{max}, \tilde{B}_\text{max})$, since $R (\tilde{B}_\text{max}, \tilde{B}_\text{min}) \{ \text{Contains}, \text{Covers}, \text{Equal} \}$ and $\text{Inside}(\tilde{A}_\text{min}, \tilde{B}_\text{min})$ then $\text{Inside}(\tilde{A}_\text{min}, \tilde{B}_\text{max})$ (2). Thus, there is contradiction among (1) and (2).
Table 4. (Continued.)

Rule 8: Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary, if $\text{Contains}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Meet}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$ then $R(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \notin \{\text{Contains, Equal, Covers, Disjoint}\}$, and vice versa.

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary where $\text{Contains}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Meet}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$ (1). We suppose now that $R(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \in \{\text{Contains, Equal, Covers, Disjoint}\}$ (2). By considering definition 1 and $\text{Contains}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$, if $\text{Contains}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}})$ then there is a contradiction because $R(\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \in \{\text{Contains, Covers, Equal}\}$ and (1). If $\text{Equal}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}})$ then there is a contradiction because $R(\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \in \{\text{Contains, Covers, Equal}\}$ and (1). If $\text{Covers}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}})$ then $R(\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\}$ or (1) is false. Finally, if $\text{Disjoint}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}})$ then there is a contradiction because $R(\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \in \{\text{Contains, Covers, Equal}\}$ and (1). Thus, (2) cannot be true.

Rule 9: Let $\tilde{A}$ and $\tilde{B}$ two regions with broad boundary, if $\text{Contains}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Covers}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$ then $R(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \notin \{\text{Meet, Disjoint}\}$, and vice versa.

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary where $\text{Contains}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Covers}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$. We suppose now that $\text{Disjoint}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}})$ or $\text{Meet}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}})$ (1). By considering definition 1 and $\text{Contains}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$, since $R(\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \in \{\text{Contains, Covers, Equal}\}$ then $\text{Contains}(\hat{A}_{\text{max}}, \hat{B}_{\text{min}})$ (2). In addition, since $\text{Covers}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}})$ and (1) then $R(\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\}$. Thus, there is a contradiction with definition 1.

Rule 10: Let $\tilde{A}$ and $\tilde{B}$ two regions with broad boundary, if $\text{Contains}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Equal}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$ then $R(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \notin \{\text{Contains, Covers, Disjoint, Meet, Overlap}\}$, and vice versa.

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary where $\text{Contains}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Equal}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$ (1). We suppose now that $R(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \in \{\text{Contains, Covers, Disjoint, Meet, Overlap}\}$ (2). If (2) then $R(\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.
Rule 11: Let \( \tilde{A} \) and \( \tilde{B} \) two regions with broad boundary, if 
\[
\text{Contains} (\tilde{A}, \tilde{B}) \quad \text{and} \quad \text{Covered by} (\tilde{A}, \tilde{B})
\]
then \( R(\tilde{A}, \tilde{B}) \) \{Covered by, Inside\}, and vice versa.

\[
\begin{align*}
\tilde{A}_\text{min} & \quad CVB(\tilde{A}_\text{min}, \tilde{B}_\text{min}) & \quad R(\tilde{A}_\text{min}, \tilde{B}_\text{max}) \in \{CVB, I\} \\
\tilde{A}_\text{max} & \quad - & \quad C(\tilde{A}_\text{max}, \tilde{B}_\text{max})
\end{align*}
\]

Proof: Let \( \tilde{A} \) and \( \tilde{B} \) two simple regions with broad boundary where \text{Contains} (\tilde{A}, \tilde{B}) \quad \text{and} \quad \text{Covered by} (\tilde{A}, \tilde{B}) (1). We suppose now that \( R(\tilde{A}_\text{min}, \tilde{B}_\text{max}) \) \{Covered by, Inside\} then \( R(\tilde{A}_\text{min}, \tilde{B}_\text{max}) \) \{Contains, Covers, Disjoint, Meet, Overlap, Equal\} (2). If (2) then \( R(\tilde{B}_\text{max}, \tilde{B}_\text{min}) \) \{Contains, Covers, Equal\} or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.

Rule 12: Let \( \tilde{A} \) and \( \tilde{B} \) two regions with broad boundary, if 
\[
\text{Contains} (\tilde{A}, \tilde{B}) \quad \text{and} \quad \text{Overlap} (\tilde{A}, \tilde{B})
\]
then \( R(\tilde{A}, \tilde{B}) \) \{Covered by, Inside, Overlap\}, and vice versa.

\[
\begin{align*}
\tilde{A}_\text{min} & \quad O(\tilde{A}_\text{min}, \tilde{B}_\text{min}) & \quad R(\tilde{A}_\text{min}, \tilde{B}_\text{max}) \in \{CVB, I, O\} \\
\tilde{A}_\text{max} & \quad - & \quad C(\tilde{A}_\text{max}, \tilde{B}_\text{max})
\end{align*}
\]

Proof: Let \( \tilde{A} \) and \( \tilde{B} \) two simple regions with broad boundary where \text{Contains} (\tilde{A}, \tilde{B}) \quad \text{and} \quad \text{Overlap} (\tilde{A}, \tilde{B}) (1). We suppose now that \( R(\tilde{A}_\text{min}, \tilde{B}_\text{max}) \) \{Contains, Covers, Disjoint, Meet, Overlap, Equal\} (2). If (2) then \( R(\tilde{B}_\text{max}, \tilde{B}_\text{min}) \) \{Contains, Covers, Equal\} or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.

Rule 13: Let \( \tilde{A} \) and \( \tilde{B} \) two regions with broad boundary, if 
\[
\text{Covers} (\tilde{A}, \tilde{B}) \quad \text{and} \quad \text{Contains} (\tilde{A}, \tilde{B})
\]
then \( R(\tilde{A}, \tilde{B}) \) \{Disjoint, Meet\}, and vice versa.

\[
\begin{align*}
\tilde{A}_\text{min} & \quad C(\tilde{A}_\text{min}, \tilde{B}_\text{min}) & \quad R(\tilde{A}_\text{min}, \tilde{B}_\text{max}) \notin \{D, M\} \\
\tilde{A}_\text{max} & \quad - & \quad CV(\tilde{A}_\text{max}, \tilde{B}_\text{max})
\end{align*}
\]

Proof: Let \( \tilde{A} \) and \( \tilde{B} \) two simple regions with broad boundary where \text{Covers} (\tilde{A}, \tilde{B}) \quad \text{and} \quad \text{Contains} (\tilde{A}, \tilde{B}) (1). We suppose now that \text{Disjoint} (\tilde{A}, \tilde{B}) \text{ or } \text{Meet} (\tilde{A}, \tilde{B}) (2). If (2) then \( R(\tilde{B}_\text{max}, \tilde{B}_\text{min}) \) \{Contains, Covers, Equal\} or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.
Table 4. (Continued.)

Rule 14: Let $\hat{A}$ and $\hat{B}$ two regions with broad boundary, if $\text{Covers}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and \text{Inside}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}}) then $\text{Inside}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}})$ and vice versa.

Proof: Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary where $\text{Covers}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Inside}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$. We suppose now that $R(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \notin \text{\{Inside\}}$ (1). Additionally, $\text{Inside}$ is a transitive relation: $\text{Inside}(A, B)$ and $\text{Inside}(B, C) \rightarrow \text{Inside}(A, C)$ (2). By considering definition 1 and (2), since $R(\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \subseteq \text{\{Contains, Covers, Equal\}}$ and $\text{Inside}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$ then $\text{Inside}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}})$ (2). Thus, (1) cannot be true.

Rule 15: Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary, if $\text{Covers}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{R}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}}) \in \text{\{Disjoint, Meet\}}$ then $\text{R}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \notin \text{\{Contains, Covers, Disjoint, Equal\}}$ and vice versa.

Proof: Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary where $\text{Covers}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{R}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}}) \in \text{\{Disjoint, Meet\}}$ (1). We suppose now that $\text{R}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \in \text{\{Contains, Covers, Disjoint, Equal\}}$ (2). If (2) then $\text{R}(\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \notin \text{\{Contains, Covers, Equal\}}$ or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.

Rule 16: Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary, if $\text{Covers}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{R}(\hat{A}_{\text{min}}, \hat{B}_{\text{min}}) \in \text{\{Equal, Covered by\}}$ then $\text{R}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \notin \text{\{Covered by, Inside\}}$ and vice versa.

Proof: Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary where $\text{Covers}(\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{R}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \in \text{\{Equal, Covered by\}}$ (1). We suppose now that $\text{R}(\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \notin \text{\{Covered by, Inside\}}$ (2). If (2) then $\text{R}(\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \notin \text{\{Contains, Covers, Equal\}}$ or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.
Table 4. (Continued.)

| Rule 17: Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary, if $\text{Covers}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}})$ and $\text{Overlap}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}})$ then $R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \in \{\text{Covered by, Inside, Overlap}\}$, and vice versa. |
| | \[
| \tilde{A}_{\text{min}} & O(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) & R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \in \{\text{CVB, I, O}\} \\
| \tilde{A}_{\text{max}} & - & \text{CV}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) \\
| |

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary where $\text{Covers}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}})$ and $\text{Overlap}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}})$ (1). We suppose now that $R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \notin \{\text{Covered by, Inside, Overlap}\}$ (2). If (2) then $R(\tilde{B}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.

| Rule 18: Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary, if $\text{Meet}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}})$ and $\text{Meet}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}})$ then $\text{Meet}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}})$ and $\text{Meet}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}})$, and vice versa. |
| | $\tilde{A}_{\text{min}}$ & $M(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}})$ & $M(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}})$ \\
| | $\tilde{A}_{\text{max}}$ & $M(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}})$ & $M(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}})$ \\
| |

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary where $\text{Meet}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}})$ and $\text{Meet}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}})$ (1). We suppose now that $R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \neq \text{Meet}(2)$ and $R(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) \neq \text{Meet}(3)$. If (2) then $R(\tilde{B}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\}$ or (1) is false. Thus, (2) cannot be true. In the same way, if (3) then there is a contradiction because $R(\tilde{B}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), there is a contradiction and (3) cannot be true.

| Rule 19: Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary, if $\text{Meet}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}})$ and $\text{Disjoint}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}})$ then $R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \in \{\text{Meet, Disjoint}\}$ and $R(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) \in \{\text{Meet, Disjoint}\}$, and vice versa. |
| | $\tilde{A}_{\text{min}}$ & $D(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}})$ & $B(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \in \{M, D\}$ \\
| | $\tilde{A}_{\text{max}}$ & $R(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) \in \{M, D\}$ & $M(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}})$ \\
| |

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary where $\text{Meet}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}})$ and $\text{Disjoint}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}})$ (1). We suppose now that $R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \notin \{\text{Meet, Disjoint}\}$ (2) and $R(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Meet, Disjoint}\}$ (3). If (2) then there $R(\tilde{B}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true. In the same way, if (3) then $R(\tilde{B}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), there is a contradiction and (3) cannot be true.
Table 4. (Continued.)

| Rule 20 | Let \( \tilde{A} \) and \( \tilde{B} \) two simple regions with broad boundary, if \( \text{Overlap} (\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) \) then \( R(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Equal, Inside, Covered by}\} \) (2), and vice versa.

\[
\begin{array}{ccc}
\tilde{A}_{\text{min}} & \tilde{B}_{\text{min}} & \tilde{B}_{\text{max}} \\
\tilde{A}_{\text{max}} & R(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) & \notin \{E, I, CVB\} \\
& O(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) & \\
\end{array}
\]

**Proof:** Let \( \tilde{A} \) and \( \tilde{B} \) two simple regions with broad boundary where \( \text{Overlap} (\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) \). According to definition 1, any region with broad boundary \( \tilde{A} \) should respect the principal following condition: \( \text{Equal}(\tilde{A}_{\text{max}}, \tilde{A}_{\text{min}}), \text{Contains}(\tilde{A}_{\text{max}}, \tilde{A}_{\text{min}}) \) or \( \text{Covers}(\tilde{A}_{\text{max}}, \tilde{A}_{\text{min}}) \) (1). We suppose now that \( R(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Equal, Inside, Covered by}\} \) (2). By considering definition 1, if (1) and (2) then \( R(\tilde{B}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\} \). Thus, there is a contradiction with definition 1.

| Rule 21 | Let \( \tilde{A} \) and \( \tilde{B} \) two simple regions with broad boundary, if \( \text{Overlap}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) \) and \( \text{Contains}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) \) then \( R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) \in \{\text{Overlap, Inside, Covered by}\} \) and \( \text{Contains}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) \), and vice versa.

\[
\begin{array}{ccc}
\tilde{A}_{\text{min}} & \tilde{B}_{\text{min}} & \tilde{B}_{\text{max}} \\
\tilde{A}_{\text{max}} & C(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) & R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \in \{O, I, CVB\} \\
& C(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) & O(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) \\
\end{array}
\]

**Proof:** Let \( \tilde{A} \) and \( \tilde{B} \) two simple regions with broad boundary where \( (\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) \) and \( \text{Contains}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) \) (1). We suppose now that \( R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{max}}) \notin \{\text{Overlap, Inside, Covered by}\} \) (2) and \( R(\tilde{A}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \text{Contains} \) (3). By considering definition 1 and \( \text{Contains}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) \), if (2) then \( R(\tilde{B}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\} \) or (1) is false. Thus, (2) cannot be true because there is a contradiction. In the same way, if (3) then \( R(\tilde{B}_{\text{max}}, \tilde{B}_{\text{min}}) \notin \{\text{Contains, Covers, Equal}\} \) or (1) is false. By considering definition 1 and \( \text{Contains}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) \), (3) cannot be true because there is also a contradiction.

| Rule 22 | Let \( \tilde{A} \) and \( \tilde{B} \) two simple regions with broad boundary, if \( \text{Overlap}(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) \) and \( R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) \in \{\text{Overlap, Meet}\} \) then \( R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) \in \{O, I, CVB\} \) and \( \text{Contains}(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) \), and vice versa.

\[
\begin{array}{ccc}
\tilde{A}_{\text{min}} & \tilde{B}_{\text{min}} & \tilde{B}_{\text{max}} \\
\tilde{A}_{\text{max}} & R(\tilde{A}_{\text{min}}, \tilde{B}_{\text{min}}) \in \{O, CV, C\} & O(\tilde{A}_{\text{max}}, \tilde{B}_{\text{max}}) \\
\end{array}
\]
Rule 23: Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary where $Overlap$ ($\hat{A}_{\max}$, $\hat{B}_{\max}$) and $R$ ($\hat{A}_{\min}$, $\hat{B}_{\min}$) $\in \{Overlap, Meet\}$ (1). We suppose now that $R$ ($\hat{A}_{\min}$, $\hat{B}_{\max}$) $\notin \{Overlap, Inside, Covered by\}$ (2) $R(\hat{A}_{\max}, \hat{B}_{\min}) \notin \{Overlap, Covers, Contains\}$ (3). If (2) then $R(\hat{B}_{\max}, \hat{B}_{\min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1, there is a contradiction and (2) cannot be true. In the same way, if (3) then $R(\hat{B}_{\max}, \hat{B}_{\min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1, there is a contradiction and (3) cannot be true.

$$
\begin{array}{ccc}
\hat{A}_{\min} & \hat{A}_{\max} & \hat{B}_{\min} \\
E(\hat{A}_{\min}, \hat{B}_{\min}) & R(\hat{A}_{\max}, \hat{B}_{\min}) & \in \{I, CVB\} \\
\end{array}
$$

Rule 24: Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary where $Overlap$ ($\hat{A}_{\max}$, $\hat{B}_{\max}$) and $Equal$ ($\hat{A}_{\min}$, $\hat{B}_{\min}$) (1). We suppose now that $R(\hat{A}_{\min}, \hat{B}_{\max}) \notin \{Inside, Covered by\}$ (2) $R(\hat{A}_{\max}, \hat{B}_{\min}) \notin \{Covers, Contains\}$ (3). If (2) then $R(\hat{B}_{\max}, \hat{B}_{\min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and $Equal$ ($\hat{A}_{\min}$, $\hat{B}_{\min}$), there is a contradiction and (2) cannot be true. In the same way, if (3) then $R(\hat{B}_{\max}, \hat{B}_{\min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and $Equal$ ($\hat{A}_{\min}$, $\hat{B}_{\min}$) (3) cannot be true because there is also a contradiction.

$$
\begin{array}{ccc}
\hat{A}_{\min} & \hat{A}_{\max} & \hat{B}_{\min} \\
I(\hat{A}_{\min}, \hat{B}_{\min}) & R(\hat{A}_{\max}, \hat{B}_{\min}) & \in \{C, CV, O\} \\
\hat{B}_{\max} & \hat{B}_{\max} & \hat{B}_{\max} \\
I(\hat{A}_{\max}, \hat{B}_{\min}) & O(\hat{A}_{\max}, \hat{B}_{\min}) & \\
\end{array}
$$

Table 4. (Continued.)
Table 4. (Continued.)

| Rule 25: Let \( \hat{A} \) and \( \hat{B} \) two simple regions with broad boundary, if \( \text{Overlap} (\hat{A}_{\max}, \hat{B}_{\max}) \) and \( \text{Covers} (\hat{A}_{\min}, \hat{B}_{\min}) \) then \( \hat{A}_{\min} \in \{\text{Inside, Covered by, Overlap}\} \) and \( \hat{A}_{\max} \in \{\text{Covers, Contains, Equal}\} \) and vice versa. |
| \( \hat{A}_{\min} \left[ CV(\hat{A}_{\min}, \hat{B}_{\min}) \quad R(\hat{A}_{\min}, \hat{B}_{\min}) \in \{I, CVB, O\} \right] \) |
| \( \hat{A}_{\max} \left[ R(\hat{A}_{\max}, \hat{B}_{\min}) \in \{CV, C\} \quad O(\hat{A}_{\max}, \hat{B}_{\min}) \right] \) |
| Rule 26: Let \( \hat{A} \) and \( \hat{B} \) two simple regions with broad boundary, if \( \text{Overlap} (\hat{A}_{\max}, \hat{B}_{\max}) \) and \( \text{Disjoint} (\hat{A}_{\min}, \hat{B}_{\min}) \) then \( \hat{A}_{\min} \in \{\text{Equal, Covers, Contains}\} \) and \( \hat{A}_{\max} \in \{\text{Equal, Covered by, Inside}\} \) and vice versa. |
| \( \hat{A}_{\min} \left[ D(\hat{A}_{\min}, \hat{B}_{\min}) \quad R(\hat{A}_{\min}, \hat{B}_{\min}) \notin \{E, CVB, I\} \right] \) |
| \( \hat{A}_{\max} \left[ R(\hat{A}_{\max}, \hat{B}_{\min}) \notin \{E, CVB, I\} \quad O(\hat{A}_{\max}, \hat{B}_{\min}) \right] \) |

**Proof:** Let \( \hat{A} \) and \( \hat{B} \) two simple regions with broad boundary where \( \text{Overlap} (\hat{A}_{\max}, \hat{B}_{\max}) \) and \( \text{Covers} (\hat{A}_{\min}, \hat{B}_{\min}) \) (1). We suppose now that \( R(\hat{A}_{\min}, \hat{B}_{\max}) \notin \{\text{Inside, Covered by, Overlap}\} \) (2) \( R(\hat{A}_{\max}, \hat{B}_{\min}) \notin \{\text{Covers, Contains}\} \) (3). If (2) then \( R(\hat{B}_{\max}, \hat{B}_{\min}) \notin \{\text{Contains, Covers, Equal}\} \) or (1) is false. By considering definition 1 and (1), (2) cannot be true because there is a contradiction. In the same way, if (3) then \( R(\hat{B}_{\max}, \hat{B}_{\min}) \notin \{\text{Contains, Covers, Equal}\} \) or (1) is false. By considering definition 1 and (1), (3) cannot be true because there is also a contradiction.

| Rule 27: Let \( \hat{A} \) and \( \hat{B} \) two simple regions with broad boundary, if \( \text{Overlap} (\hat{A}_{\max}, \hat{B}_{\max}) \) and \( \text{Covered by} (\hat{A}_{\min}, \hat{B}_{\min}) \) then \( \hat{A}_{\min} \in \{\text{Inside, Covered by, Overlap}\} \) and \( \hat{A}_{\max} \in \{\text{Covers, Contains, Overlap}\} \) and vice versa. |
| \( \hat{A}_{\min} \left[ CVB(\hat{A}_{\min}, \hat{B}_{\min}) \quad R(\hat{A}_{\min}, \hat{B}_{\max}) \in \{I, CVB\} \right] \) |
| \( \hat{A}_{\max} \left[ R(\hat{A}_{\max}, \hat{B}_{\min}) \in \{CV, C, O\} \quad O(\hat{A}_{\max}, \hat{B}_{\max}) \right] \) |
there is a contradiction. In the same way, if (3) then $R \overset{\text{Covers}}{\ni} \overset{\text{Equal}}{\ni} \overset{\text{Inside}}{\ni}$ (3). If (2) then $R \overset{\text{B}_{\text{max}}, \text{B}_{\text{min}}}{} \not\ni \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), (2) cannot be true because there is a contradiction. In the same way, if (3) then $R \overset{\text{B}_{\text{max}}, \text{B}_{\text{min}}}{} \not\ni \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), (3) cannot be true because there is also a contradiction.

**Proof:** Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary where $\text{Overlap} (\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Covered} by (\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$ (1). We suppose now that $R (\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \not\ni \{\text{Inside, Covered by}\}$ (2) $R(\hat{A}_{\text{max}}, \hat{B}_{\text{min}}) \not\ni \{\text{Covers, Contains, Overlap}\}$ (3). If (2) then $R (\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \not\ni \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), (2) cannot be true because there is a contradiction. In the same way, if (3) then $R (\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \not\ni \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), (3) cannot be true because there is also a contradiction.

**Rule 28:** Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary, if $\text{Contains} (\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Disjoint} (\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$ then $R (\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \not\ni \{\text{Contains, Covers, Equal}\}$ and $\text{Contains} (\hat{A}_{\text{max}}, \hat{B}_{\text{min}})$, and vice versa.

**Proof:** Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary where $\text{Contains} (\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Disjoint} (\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$ (1). We suppose now that $R (\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \in \{\text{Contains, Covers, Equal}\}$ (2) and $R(\hat{A}_{\text{min}}, \hat{B}_{\text{min}}) \not\ni \{\text{Contains}\}$ (3). If (2) then $R (\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \not\ni \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), (2) cannot be true because there is a contradiction. In the same way, if (3) then $R (\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \not\ni \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), (3) cannot be true because there is also a contradiction.

**Rule 29:** Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary, if $\text{Covers} (\hat{A}_{\text{max}}, \hat{B}_{\text{max}})$ and $\text{Covers} (\hat{A}_{\text{min}}, \hat{B}_{\text{min}})$ (2). By considering definition 1 and (1), (2) cannot be true because there is a contradiction. In the same way, if (3) then $R (\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \not\ni \{\text{Contains, Covers, Equal}\}$ or (1) is false. By considering definition 1 and (1), (3) cannot be true because there is also a contradiction.

**Rule 30:** Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary, if Rule 29 and $\text{Covers} (\hat{A}_{\text{min}}, \hat{B}_{\text{max}})$ Then $R (\hat{A}_{\text{max}}, \hat{B}_{\text{min}}) \in \{\text{Covers}\}$, and vice versa.

**Proof:** Let $\hat{A}$ and $\hat{B}$ two simple regions with broad boundary where Rule 29 and $R (\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \in \{\text{Covers}\}$ (1). We suppose now that $R (\hat{A}_{\text{min}}, \hat{B}_{\text{max}}) \not\ni \{\text{Covers}\}$ (2). By considering definition 1 and (1), if (2) then $R (\hat{B}_{\text{max}}, \hat{B}_{\text{min}}) \not\ni \{\text{Contains, Covers, Equal}\}$ or (1) is false. Thus, (2) cannot be true because there is also a contradiction.
Table 4. (Continued.)

<table>
<thead>
<tr>
<th>Rule 31</th>
<th>$\tilde{A}$ and $\tilde{B}$ two simple regions with broad boundary, if Rule 29 and $R(\tilde{A}<em>{\min}, \tilde{B}</em>{\max}) \in {\text{Inside}}$, then $R(\tilde{A}<em>{\max}, \tilde{B}</em>{\min}) \in {\text{Contains}}$, and vice versa.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\begin{array}{ccc}A_{\min} &amp; \tilde{B}<em>{\min} &amp; \tilde{B}</em>{\max} \ \tilde{A}<em>{\min} &amp; CV(A</em>{\min}, B_{\min}) &amp; R(A_{\min}, B_{\max}) \in {I} \ \tilde{A}<em>{\max} &amp; R(A</em>{\max}, B_{\min}) \in {C} &amp; CV(A_{\max}, B_{\max}) \end{array}$</td>
</tr>
</tbody>
</table>